7th International Tournament of Young Mathematicians

QUIZ

1.5 hours

- Each team (high school students only) is gathered in a separate room and works together. Written materials, electronics, literature or other sources are forbidden during the quiz, as well as any external help. Only brochures of the ITYM and paper language dictionaries are allowed.
- A solution for each of the 10 problems should be written **separately**.
- Indicate the **problem number** and page numbers on every solution.
- Please **don't** mention your country, team or other names anywhere.

Good luck!

Problem 1. A Laser Machine

1. Given a set of points in the real plane, when is it *convex*? **(1 point)**

2. In the case where P is a circle containing the center of the disk, σ is the diameter of P and w is negligible, find the minimal number of beams needed to determine σ with an error $\varepsilon \leq \frac{1}{2}$. **(4 points)**

3. Let P be a circle not necessarily containing the center of the disk. Suppose that the machine is broken and can only emit beams parallel to the x-axis with a non-negligible w. The engineer would like to determine the diameter of P with only two shots. What is the smallest possible ε he can guarantee regardless of the position of P? **(5 points)**

Problem 2. Maximal Minimal Triangles

1. What is an *affine transformation* of the plane? How does it change area of plane regions? **(2 points)**

2. What is the maximal area of a triangle inscribed into an ellipse of area 1? **(3 points)**

3. A triangle is inscribed into a parabola. The upper side of the triangle together with the corresponding part of the parabola (see Figure 1) bound a region of area 1.

a) What is the maximal possible area of the triangle? **(4 points)**

b) For which triangles the maximal area is achieved? **(1 point)**

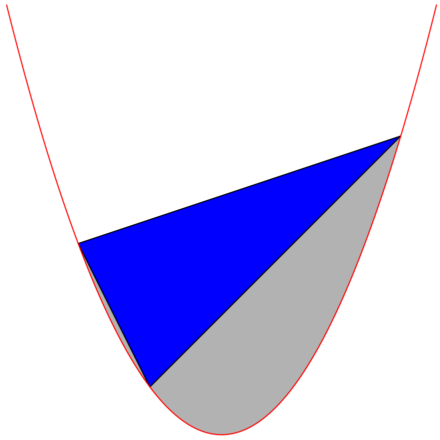

FIGURE 1. The area of the region bounded by the upper side of the triangle and the lower part of the parabola is 1.

Problem 3. Coloured Circles

1. Give a definition of a closed *d-dimensional ball* of radius 1. **(1 point)**

2. Consider the game on the real line with all segments having length 1.

a) Show that $C_1(n,k) \leq 2k-1$ for any natural n and k. **(2 points)**

b) Suppose that k is even and $n \geq 2k$. Prove that $C_1(n,k) \geq \frac{3}{2}k$. **(3 points)**

3. Suppose now that Carl is greedy and he applies a new colour only when he has to (if Clara puts a segment and Carl can colour it with one of the colours he has used previously, then he will certainly do it). Denote by $GC_1(n,k)$ the minimum number of colours that Carl would need to successfully colour n segments put by Clara on a line, provided that no point is covered by more than k segments. Show that $GC_1(n,k) = 2k-1$. **(4 points)**

Problem 4. Simple Paths in Grids

1. a) Give definitions of a *connected* graph and a *complete* graph. **(1 point)**

b) Show that a graph with n vertices and n edges must have a cycle. **(1 point)**

c) A graph is called *bipartite* if the set of its vertices can be divided into two disjoint subsets U and V such that every edge connects a vertex in U to one in V. Show that any $n \times n$ grid is a bipartite graph. **(1 point)**

2. Find and prove a formula for $D_n(2)$. **(3 points)**

3. Let $G = (V, E)$ be a graph. A *perfect matching* of G is a subset of its edges $P \subseteq E$ such that every vertex of G is an endpoint of **exactly** one edge in P.

Find the number $\rho(G)$ of perfect matchings of G in the following cases:

a) G is a complete graph on $2n$ vertices. **(2 points)**

b) $G = (U, V, E)$ is a *complete* bipartite graph on m and n vertices. In other words, the set of vertices of G consists of two disjoint sets U and V of size m and n, respectively, and E is the set of edges connecting every vertex in U with all vertices in V. **(2 points)**

Problem 5. A Recursive Sequence

1. Give a definition that a real sequence (x_n) *converges* to a point $r \in \mathbb{R}$. **(1 point)**

2. Consider two sets $A \subset A' \subset [0,1]$. Let (u_n) and (u'_n) be recursive sequences for A and A', respectively, such that $u_0 = u'_0$. Prove that $u_n \leq u'_n$ for all $n \in \mathbb{N}$. **(3 points)**

3. Take $A = \left\{ \frac{k}{2^m} \mid k, m \in \mathbb{N} \cup \{0\} \text{ and } k \leq 2^m \right\}$.

a) Find $\mathscr{L}_0(A)$. **(2 points)**
b) Find $\mathscr{L}_\infty(A)$. **(4 points)**

Problem 6. Critical Points

1. Consider a function $f :]a, b[\to \mathbb{R}$ defined on an open interval $]a, b[$.

a) Give a definition of the derivative of f at a point $x_0 \in]a, b[$. Is it true that if f is differentiable at x_0 then it must be continuous at x_0? **(1 point)**

b) Give a geometric interpretation of the derivative. **(1 point)**

2. a) Show that the function $F(x) = \sum_{k=1}^{2016} \frac{1}{x-k}$ doesn't have critical points. **(1 point)**

b) Show that the following function

$$H(x) = \frac{1}{x-1} - \frac{1}{x-2} + \frac{1}{x-3}$$

has at most 4 critical points in its domain $\mathbb{R} \setminus \{1, 2, 3\}$. **(1 point)**

c) Prove that H has at least 2 critical points. **(3 points)**

3. Consider the function $F(x) = \sum_{k=1}^{2016} \frac{1}{x-k}$ as a ratio

$$F(x) = \frac{P(x)}{Q(x)}$$

of two polynomials $P(x)$ and $Q(x)$ of degrees 2015 and 2016 respectively. Prove that the polynomial $P(x)$ has 2015 distinct real roots. **(3 points)**

Problem 7. Chain Stores

1. Let f be a piecewise continuous function on the interval $[0,1]$. Give a geometric interpretation of the integral $\int_0^1 f(x)dx$. **(1 point)**

2. Let $n > 2$ be a natural number. Compare the following integrals:

$$I_1 = \int_0^1 dist(x, S_1)dx \quad \text{and} \quad I_2 = \int_0^1 dist(x, S_2)dx\,,$$

where $S_1 = \left\{\frac{k}{n}\right\}_{k=1}^{n-1}$ and $S_2 = \left\{\frac{k}{n} + \frac{1}{n^2}\right\}_{k=1}^{n-1}$ are sets of $n-1$ points in $[0,1]$. **(3 points)**

3. Take $f(x) = -2\,|2x-1| + 2$ and $n = 2$. Let S_{opt} be an optimal configuration of shops when the merchant builds the shops all at once, and let $\tilde{S}_{opt}$ be that when the merchant builds the shops one after another.

a) Find S_{opt} and $\tilde{S}_{opt}$. **(4 points)**

b) Find the ratio between $\int_0^1 dist(x, S_{opt})f(x)\,dx$ and $\int_0^1 dist(x, \tilde{S}_{opt})f(x)\,dx$. **(2 points)**

Problem 8. A Diophantine Equation

1. a) Give a definition of a *multiplicative* function (in number theory). **(1 point)**

b) Let n be a natural number and denote by $\tau(n)$ the number of its natural divisors, including 1 and n. Give a formula for $\tau(n)$. **(1 point)**

2. Denote by $g(n)$ the number of solutions of the equation

$$xyz = n$$

in positive integers x, y, z. Prove that $g(n) < n^{\frac{1}{2015}}$ for sufficiently large n. **(3 points)**

3. A *Carmichael number* is a **composite** number n such that

$$n \text{ divides } x^{n-1} - 1$$

for all integers $1 < x < n$ which are relatively prime to n.

a) Prove that all Carmichael numbers are odd. **(2 points)**

b) A natural n is called *square-free* if it is not divisible by any square k^2, where $k > 1$ is an integer. Prove that a Carmichael number must be square-free. **(3 points)**

Problem 9. Framing Matrices

1. a) Give a definition of a *vector space* over a field K. **(1 point)**
b) When does a system of vectors is *linearly independent* over K? **(1 point)**

2. Find the number of framing pairs (A, B) such that:

- A is a diagonal 2×2 matrix of order 6 and
- $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **(4 points)**

3. Three vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ are randomly chosen with the condition that every coordinate of each vector is a real number from the interval $[0, 1]$. What is the probability that these three vectors are linearly independent over $\mathbb{R}$? **(4 points)**

Problem 10. Polynomial Groups

1. Give definitions of a *group* and an *abelian group.* **(1 point)**

2. Let $p \geq 2$ be a prime number.

a) Prove that any group G of order p^2 is abelian. The *order* of a group is the number of its elements. **(3 points)**

Indication. Consider the action of the group G on its elements by conjugation (for any $h \in G$ one has a map $f_h : G \to G$, $g \mapsto hgh^{-1}$). Present G as a disjoint union of orbits $G(g) = \left\{hgh^{-1} \mid h \in G\right\}$. Investigate how many elements there are in the center of the group G, namely in the subgroup $Z = \left\{g \in G \;\middle|\; hgh^{-1} = g \text{ for all } h \in G\right\}$.

b) Show that the polynomial group $G_3(p)$ is abelian. **(1 point)**

3. Let $p \geq 3$ be an odd prime number. Consider the following set of upper-triangular matrices over the field $\mathbb{F}_p$:

$$H_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}.$$

a) Prove that H_p is a group. It is called the *Heisenberg group* over $\mathbb{F}_p$. **(2 points)**

b) Show that the group H_p is two-generator. **(3 points)**