Team: Russia 2

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Problem №7
An Experiment

Abstract

In this paper we have studied some properties of n - typles in the set F_2 with the given addition rules and got some results. For example, we have investigated some interesting cases from which we can get a balance tuple. We proved that there can not be balanced typles in many cases for more than 1 iteration in a raw. We suggested the use of the Pascal's triangle in that problem what can be very helpful in further investigations. Also we considered new problem with new-defined operation in the question 2.

1st Question

Notation. It is obviuos that n - typles (n is even) with alternation of 1 and 0, for example:

$(1, 0, 1, 0, 1, 0, ...1, 0)$

Dont have any interest because after iteration they turn into typle of zeros from which we can not get any 1 in typle according to our definition of α and addition rules.

Definition. Period of the tuple is the subtyple which we can met more than 1 time in our typle. This subtyples should not intersect.

Statement 1. If there is a period in the typle and $n \vdots 4$: period of $(0, 0, 1, 1)$,$(0, 0, 1)$,$(1, 1, 0)$ or $(1, 1, 0, 0)$ then on first iteration we get balanced typle.

We can consider only period of our typle because the first element after our period is equal to the last in the first period:

$(0, 0, 1, 1) \to (0, 1, 0, 1)$

$(1, 1, 0, 0) \to (0, 1, 0, 1)$

When there are 3 elements in a period we get that in any case our period becomes $(1, 0, 1)$, and the number of periods is still the same.

So we have that there are $\frac{n}{2}$ 1 and $\frac{n}{4}$ zeros. But as our first typle should be balanced there are exactly $\frac{n}{4}$ 1 (or $\frac{n}{4}$ zeros for second n-typle) out-of-period.

So, after first iteration we get balanced typle.

Statement 2. If there are $\frac{n}{4}$ zeros in our typle in a row (from x_1 to $x_{\frac{n}{4}}$) and $\frac{n}{4}$ 1 (from $x_{\frac{3n}{4}}$ to x_n) and alternation of 1 and 0 (alternation begins from 0) at the middle then after first iteration it is a balanced n-typle.

Our first $\frac{n}{4}$ zeros become $\frac{n}{4} - 1$ zeros and the same for last ones. Also we get 2 zeros from splice of our alternation with the first and last $\frac{n}{4}$ numbers.

There are $\frac{n}{2}$ elements in alternation what after first iteration will give us $\frac{n}{2} - 1$ of ones. And one more 1 from addition of last and first element in tuple.

Therefore we got same amount of zeros and ones.

Statement 3. If $n = 2^k$ and our typle is balanced then on n'th iteration we get $(0, 0, 0, ..., 0)$.

Consider any element in our typle. Note that we add the following elements with the coefficients from the Pascal's triangle (for l'th iteration - from l'th string of the triangle. It is a well-known fact that all coefficients at the 2^kth string are odd.

On n‘th iteration we add every element to a given one with odd coefficient:

$$x_1 + a_1x_2 + a_2x_3 + ... + a_{n-2}x_{n-1} + x_n = (x_1 + x_2 + ... + x_n) + (a_1 - 1)x_2 + (a_2 - 1)x_3 + ... + (a_{n-2} - 1)x_{n-1} = x_1 + x_2 + ... + x_n = 0$$

Theorem. If $\alpha^1(x_1, x_2, ..., x_n)$ is balanced then $\alpha^2(x_1, x_2, ..., x_n)$ can not be balanced as well excluding the case when in the second iteration all elemens go in alternation.

Consider the second iteration according to the note from the previous statement.

So we get:

$$(x_1 + x_3, x_2 + x_4, ..., x_{n-2} + x_n, x_{n-1} + x_1, x_n + x_2)$$

Consider the case when all $\frac{n}{2}$ zeros go in a row. Then we get that in first iteration there should be more than $\frac{n}{2}$ similar elements and therefore this tuple can not be balanced.

Now consider the case when some elements can go in a row. Then if there is some k zeros in a row we get that in first iteration there will be $k+1$ similar elements in a raw in the first iteration. That means that tuple can not be balanced.

Suppose that $\alpha^2(x_1, x_2, ..., x_n)$ is also balanced. Then there are exactly $\frac{n}{2}$ zeros and ones in the second iteration. That means we have exactly $\frac{n}{2}$ sums with equal elements.

2nd Question

We consider new operation - multiplication with the following rules:

$1 * 1 = 1$

$1 * 0 = 0$

$0 * 0 = 0$

The set is the same.

Statement. There can not be balanced typles 2 times in a raw and super-balanced as well.

It is easy to see that if there is exactly k zeros on some iteration then on next iteration there will be more than k zeros. And the number of zeros obviously can not decrease.

Note that in order to get balanced typles the number of ones should be greater than the number of zeros. Every splice 1 with 0 gives one more additional zero.

It is obvious that there can not be just 1 splice because our typle is retired. So the number of balanced typles should be greater or more than the number of possible slices -1.