Team: Russia 2

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Problem №6
Recurrent Sequences

Abstract

We considered all the given questions and made a research for each of them. Depending on the u_1 and u_2 we have got an understanding of some properties of the sequences. For example, we have proved the monotony for first three sequences, investigated the first one for its convergence. Also we have proved the existance of u_4 in question 4 (b).

Solution of the problem

1st Question

$$u_{n+1} = \frac{u_1^2+u_2^2+...+u_n^2}{n}$$

We consider u_{k+1} and denote by a the following sum:

$$a = u_1^2 + u_2^2 + ... + u_k^2$$

$$u_{k+1} = \frac{a}{k}$$

$$u_{k+2} = \frac{a+\frac{a^2}{k^2}}{k+1} = \frac{ak^2+a^2}{k^2(k+1)}$$

Statement 1. The following member of the sequence is greater than the previous if the previous is greater than 1.

Consider the given u_{k+2} and u_{k+1}:

$$u_{k+2} > u_{k+1}$$

$$\frac{ak^2+a^2}{k^2(k+1)} > \frac{a}{k}$$

$$\frac{k^2+a}{k^2+k} > 1$$

$$k^2 + a > k^2 + k$$

What can be possible only in case:

$$u_{k+1} = \frac{a}{k} > 1$$

From this statement obviously follows that our sequence is monotonically increasing when $u_1^2 + u_2^2 > 2$ and monotonically decreasing when $u_1^2 + u_2^2 < 2$.
Case $u_1^2 + u_2^2 = 2$ does not have any interest because for any $k \geq 3$ $u_k = 1$.

Statement 2. For any $k \geq 2$ $\quad \frac{u_{k+3}}{u_{k+2}} \geq \frac{u_{k+2}}{u_{k+1}}$.

$$\frac{u_{k+2}}{u_{k+1}} = \frac{k^2+a}{k^2+k} = \frac{(k^2+a)\left(\frac{k+2}{k}\right)}{k(k+1)\left(\frac{k+2}{k}\right)} = \frac{k^2+2k+a+\frac{2a}{k}}{(k+1)(k+2)}$$

$$\frac{u_{k+3}}{u_{k+2}} = \frac{k^2+2k+1+a+\frac{a^2}{k^2}}{k^2+2k+1+k+1} = \frac{k^2+2k+1+a+\frac{a^2}{k^2}}{(k+1)(k+2)}$$

Compare the fractions $\frac{u_{k+3}}{u_{k+2}}$ and $\frac{u_{k+2}}{u_{k+1}}$:

$$k^2 + 2k + 1 + a + \frac{a^2}{k^2} \geq k^2 + 2k + a + \frac{2a}{k}$$

$$\frac{a^2}{k^2} + 1 \geq \frac{2a}{k}$$

It is equal:

$$\left(\frac{a}{k} + 1\right)^2 \geq 0$$

From this statement follows that the distance between two nearby members of the sequence is growing when k is also growing. According to the Archimedean property we can find such m that u_m is greater than a given positive number. That means $\lim\limits_{n\to\infty}(u_n) = +\infty$.

Question 2.

$$u_{n+1} = \frac{u_1u_n + u_2u_{n-1} + u_3u_{n-2} + ... + u_nu_1}{n}$$

At first we get that the next member of sequence is:

$$u_{n+2} = \frac{u_1u_{n+1} + u_2u_n + u_3u_{n-1} + ... + u_nu_2 + u_1u_{n+1}}{n+1}$$

Then, we make the denominator of the first fraction to be equal to $n+1$:

$$u_{n+1} = \frac{u_1u_n + u_2u_{n-1} + u_3u_{n-2} + ... + u_nu_1 + \frac{u_1u_n + u_2u_{n-1} + u_3u_{n-2} + ... + u_nu_1}{n}}{n+1} = \frac{u_1u_n + u_2u_{n-1} + u_3u_{n-2} + ... + u_nu_1 + u_{n+1}}{n+1}$$

Statement 3. $\frac{u_1u_{n+1} + u_2u_n + u_3u_{n-1} + ... + u_nu_2 + u_1u_{n+1}}{n+1} > \frac{u_1u_n + u_2u_{n-1} + u_3u_{n-2} + ... + u_nu_1 + u_{n+1}}{n+1}$ when $1 < u_1 < u_2$.

Statement 4. $\frac{u_1u_{n+1} + u_2u_n + u_3u_{n-1} + ... + u_nu_2 + u_1u_{n+1}}{n+1} < \frac{u_1u_n + u_2u_{n-1} + u_3u_{n-2} + ... + u_nu_1 + u_{n+1}}{n+1}$ when $1 > u_1 > u_2$.

Both statements are obvious according simple induction by n.
Therefore we get requered conditions for the monotony of the sequence.

Question 3.

σ is a random permutation of $1, 2, ..., n$.

$$u_{n+1} = \frac{u_1u_{\sigma 1} + u_2u_{\sigma 2} + u_3u_{\sigma 3} + ... + u_nu_{\sigma n}}{n}$$

Let σ_1 be permutation of $1, 2, ..., n$ such as $\forall m \, \sigma_1(u_m) = u_{n-m+1}$. It is obvious that any σ (random) permutation can be obtained from σ_1 by changing k times the position of 2 nearby elements. When some element gets a position equal to the position of this element of result of σ, elements after this element have their positions in the decreasing order.

Let's consider the moment when positions of $u_{\sigma_1 k}$ and $u_{\sigma_1(k+1)}$ have changed. According to the definition of σ_1 we have:

$$u_{\sigma_1 k} > u_{\sigma_1(k+1)}$$

And:

$$u_ku_{\sigma_1(k)} + u_{k+1}u_{\sigma_1(k+1)} = u_ku_{n-k+1} + u_{k+1}u_{n-k}$$

$$(u_ku_{n-k} + u_{k+1}u_{n-k+1}) - (u_ku_{n-k+1} + u_{k+1}u_{n-k}) =$$

$$= u_{k+1}(u_{n-k+1} - u_{n-k}) - u_k(u_{n-k+1} - u_{n-k}) =$$

$$= (u_{k+1} - u_k)(u_{n-k+1} - u_{n-k}) > 0$$

So, we can say that with every changing of the positions of nearby σ-elements the nominator of any u_{n+1} is growing. So it is easy to see that the sequence is increasing.

Other cases can be investigated analogically.

σ is a random function $U \to U$.

Now we want to point out that there can be no special properties. We'd like to consider the case $u_1 = 1$, $u_2 = 3$.

$$u_3 = \frac{1*3+3*3}{2} = 6$$

It is obvious that u_4 can have some (more, than one) possible values. For example, $u_4 = \frac{1+3+6}{3} = 3\frac{1}{3}$ and $u_4 = \frac{1*3+3*3+6*3}{3} = 10$.

u_5 also can have different values: $\frac{1*3+3*3+6*3+3\frac{1}{3}*3}{4} = 10$ or $\frac{1+3+6+3\frac{1}{3}}{4} = 3,25 + \frac{1}{12} < 3,34$

u_6 also has some values in each case. One of them is $\frac{1+3+6+3\frac{1}{3}+10}{5} = 5.2 + \frac{1}{15}$.

We see that in every case there can be monotony or not to be (also we can say the same for other properties). So there is no regularity with the values of members.

Question 4 (b).

$$u_n = \frac{u_1 u_{n+1} + u_2 u_n + \ldots + u_{n+1} u_1}{n+1}$$

$$u_1 = u_2 = u_3 = 1$$

There is a value of $u_4$such that every element of the sequence is less than one. Let's consider first elements of this sequence:

$$u_4 = 0.2$$

$$u_5 = -0,2$$

$$u_6 = -0,6$$

$$u_7 = -2,64$$

$$u_8 = -21,12$$

$$u_9 = -71,18$$

$$u_{10} = -263,192$$

Using induction, we proved that every $u_{n+1} < u_n$. Induction base is proved ($u_{10} < u_9 < u_8 < ... < u_5$).

$$u_n = \frac{2u_{n+1} + 2u_n + 2u_{n-1} + 0.4u_{n-2} + [pairwise\, multiplications\, of\, the\, previous\, elements]}{n+1}$$

It is obvious that nominator is equal to $u_n(n+1)$. So, if $2u_n + 2u_{n-1} + 0.4u_{n-2} + [pairwise\, multiplications\, of\, the\, previous\, elements$ $u_n(n-1)$ then $2u_{n+1} < 2u_n$, $u_{n+1} < u_n$.

Pairwise multiplications of the previous elements are greater then zero. $2u_{n-1} > 2u_n$. $0,4u_{n-2} > 2u_n$. $2u_n = 2u_n$.