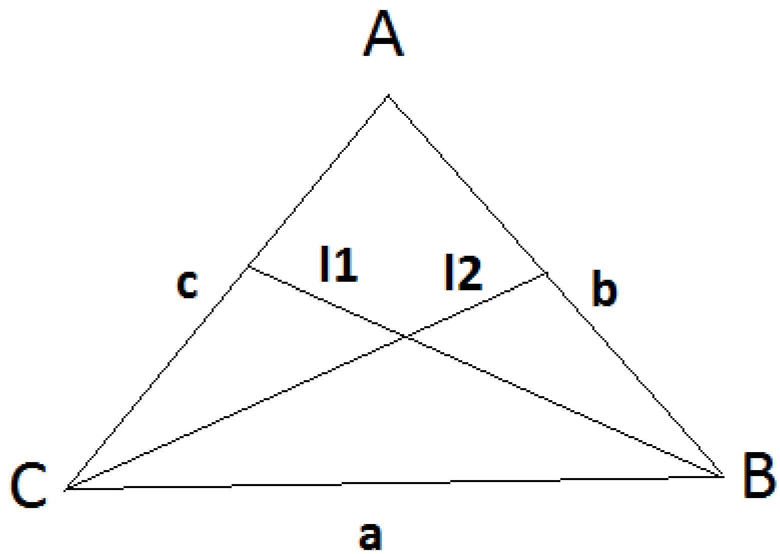
Team:Russia 2

Saint-Petersburg

Problem 4

Isosceles Triangles.

1 Prove that two internal angle bisectors of a triangle are equal if and only if the triangle is isosceles.



It is a well-known fact that the lengths of bisectors have the following representations:

$$l_1 = \frac{\sqrt{ab(a+b+c)(a+b-c)}}{(a+b)}, \qquad l_2 = \frac{\sqrt{ac(a+b+c)(a+c-b)}}{a+c}.$$

So $\frac{\sqrt{ab(a+b+c)(a+b-c)}}{(a+b)} = \frac{\sqrt{ac(a+b+c)(a+c-b)}}{a+c} \quad \Leftrightarrow \quad b(a+b-c)(a+c)^2 = c(a+c-b)(a+b)^2 \quad \Leftrightarrow$

$\Leftrightarrow \quad a^3b + 2a^2bc + abc^2 + a^2b^2 + 2ab^2c + b^2c^2 - a^2bc - 2abc^2 - bc^3 = a^3c + 2a^2bc + ab^2c + a^2c^2 + 2abc^2 + b^2c^2 -$

$a^2bc - 2ab^2 - 2ab^2c - b^3c \quad \Leftrightarrow \quad a^3(b-c) + a^2(b-c)(b+c) + 3abc(b-c)(b+c) + bc(b-c)(b+c) = 0 \quad \Leftrightarrow$

$\Leftrightarrow \quad (b-c)(a^3 + a^2(b+c) + 3abc(b+c) + bc(b+c)) = 0$

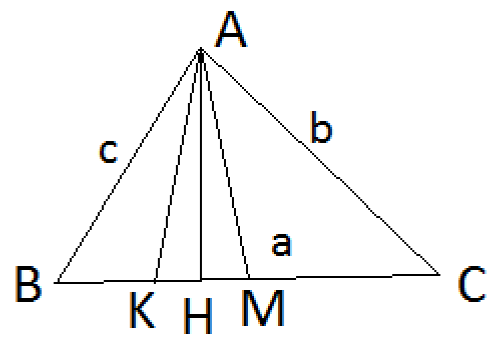
We have this:

$$\left[\begin{array}{l} b-c=0 \\ a^3 + a^2(b+c) + 3abc(b+c) + bc(b+c) = 0. \end{array}\right.$$

From it obviously follows that $b = c$.

2 Prove the two symmedians of a triangle are equal if and only if the triangle is isosceles.

Proposition 2.1. *The symmedian through a vertex of a triangle divides the opposite side in proportion 2^{th} powers of the sides of the triangle.*

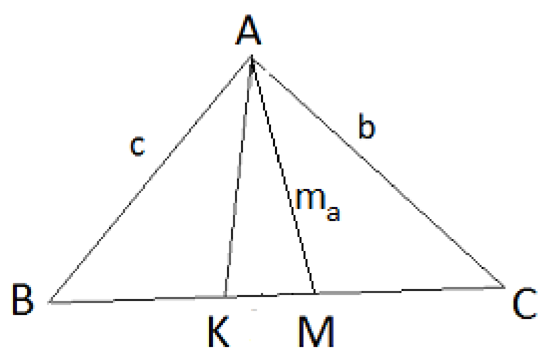


Proof.
Let AM be a median, AN be a bisector and AK be a symmedian.

According to the theorem of sines $\frac{AB}{BM} = \frac{\sin(AMB)}{\sin(BAM)}$ and $\frac{AB}{BK} = \frac{\sin(AKB)}{\sin(BAK)}$,

so $\frac{AB}{BM} * \frac{AB}{BK} = \frac{\sin(AMB)*\sin(AKB)}{\sin(BAM)*\sin(BAK)} = \frac{\sin(AMC)*\sin(AKC)}{\sin(CAK)*\sin(CAM)} = \frac{AC}{KC} * \frac{AC}{CM}$. So $\frac{AB^2}{AC^2} = \frac{BM*BK}{KC*CM} = \frac{BK}{KC}$ □

Now, we are finding the length of a symmedian.



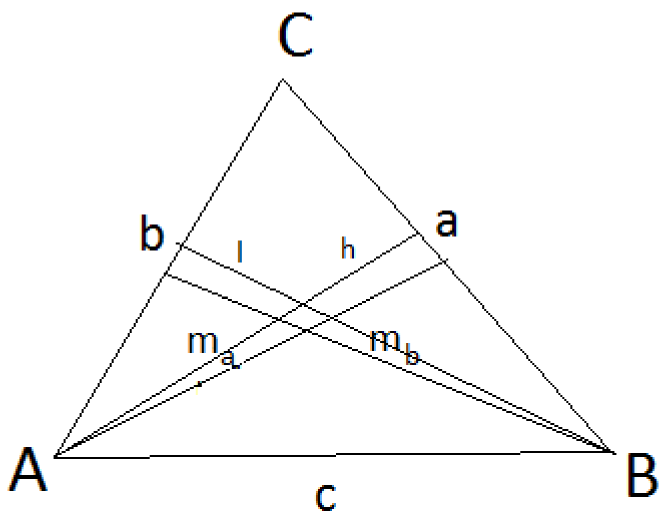
Let AM be a median, AH be a height, AK be a symmedian.

Bu Stuart's theorem $AK^2 = c^2\frac{KC}{a} + b^2\frac{BK}{a} - BK * KC$.

Since $\frac{BK}{KC} = \frac{c^2}{b^2}$ $\frac{BK}{c^2} = \frac{KC}{b^2} = \frac{a}{c^2+b^2}$ so $BK = \frac{ac^2}{c^2+b^2}$ and $KC = \frac{ab^2}{b^2+c^2}$.

$AK^2 = \frac{c^2b^2}{b^2+c^2} + \frac{b^2c^2}{b^2+c^2} - \frac{a^2b^2c^2}{(c^2+b^2)^2} = \frac{b^2c^2}{(b^2+c^2)^2}(2b^2 + 2c^2 - a^2) = \frac{b^2c^2}{(b^2+c^2)^2}(m_a^2)$

Consider the triangle ABC with medians m_a and m_b.



Consider their symmedians.

$$l = h \quad \Leftrightarrow \quad \frac{acm_b}{a^2+c^2} = \frac{bcm_a}{b^2+c^2} \quad \Leftrightarrow \quad am_b(b^2+c^2) = bm_a(a^2+c^2) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad a\sqrt{2a^2+2c^2-b^2}(b^2+c^2) = b\sqrt{2b^2+2c^2-a^2}(a^2+c^2) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad a^2(2a^2+2c^2-b^2)(b^4+c^4+2b^2c^2) = b^2(2b^2+2c^2-a^2)(a^4+c^4+2a^2c^2) \quad \Leftrightarrow \quad 2a^4b^4+2a^4c^4+4a^4b^2c^2+$$

$$2a^2c^6+3a^2b^2c^4-a^2b^6 = 2a^4b^4+2b^4c^4+4a^2b^4c^2+2b^2c^6+3a^2b^2c^4-a^6b^2 \quad \Leftrightarrow \quad 2a^4c^4+4a^4b^2c^2+2a^2c^6+a^6b^2$$

$$= 2b^4c^4+4a^2b^4c^2+2b^2c^6+a^2b^6 \quad \Leftrightarrow \quad (a^2-b^2)(2c^4(a^2+b^2)+4a^2b^2c^2+2c^6+a^2b^2(a^2+b^2)) = 0.$$

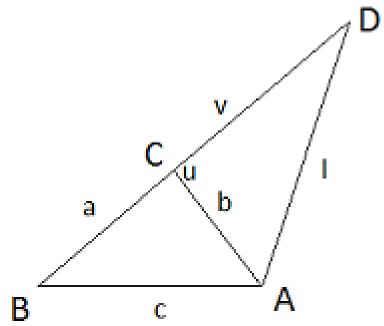
We have this:

$$\left[\begin{array}{l} a-b=0 \\ a+b=0 \\ 2c^4(a^2+b^2)+4a^2b^2c^2+2c^6+a^2b^2(a^2+b^2)=0. \end{array}\right.$$

From it obviously follows that $a = b$.

3 Is it true that two external angle bisectors of a triangle are equal if and only if the triangle is isosceles?

Consider a triangle ABC. Let AD be an external bisector.

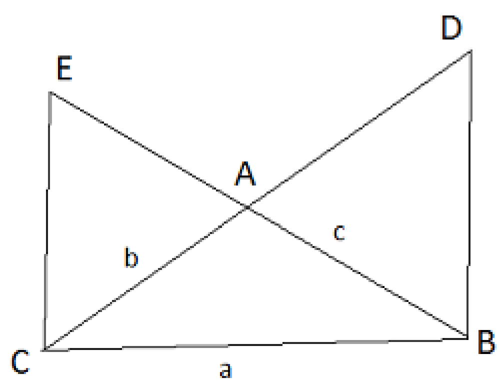


By the property of external bisector $\frac{v}{u} = \frac{b}{c}$ so $v = \frac{ab}{c-b}$ and $u = \frac{ac}{c-b}$.

By the Stuart's theorem $c^2v + l^2a - auv = b^2u$

so $l^2 = \frac{a^2c}{(c-b)^2} - \frac{b}{c-b}c^2 + \frac{c}{c-b}b^2 = \frac{a^2bc}{(c-b)^2} - \frac{cb(c-b)}{c-b} = \frac{a^2bc-cb(c-b)^2}{(c-b)^2} = \frac{cb(a^2-(c-b)^2)}{(c-b)^2} = \frac{cb(a-c+b)(a+c-b)}{(c-b)^2}$.

Consider a triangle ABC with exteranl bisectors BD and CE.



$BD = CE \quad \Leftrightarrow \quad \frac{ab\sqrt{a+c-b}\sqrt{b+c-a}}{b-a} = \frac{ac\sqrt{a+b-c}\sqrt{c+b-a}}{c-a} \quad \Leftrightarrow \quad \frac{b\sqrt{a+c-b}}{b-a} = \frac{c\sqrt{a+b-c}}{c-a} \quad \Leftrightarrow \quad \frac{(a+c-b)b^2}{(b-a)^2} = \frac{(a+b-a)c^2}{(c-a)^2} \quad \Leftrightarrow$

$\Leftrightarrow \quad (ab^2 + cb^2 - b^3)(c-a)^2 = (b-a)^2(ac^2 + bc^2 - c^3) \quad \Leftrightarrow \quad a^3b^2 - a^2b^3 - a^2b^2c + 2ab^3c - b^3c^2 + b^2c^3 = a^3c^2 - a^2bc^2 - a^2c^3 + 2abc^3 + b^3c^2 - b^2c^3 \quad \Leftrightarrow \quad a^3(b-c)(b+c) - a^2(b-c)(b^2+bc+c^2) - a^2bc(b-c) + 2abc(b-c)(b+c) - 2b^2c^2(b-c) = 0 \quad \Leftrightarrow \quad (b-c)(a^3(b+c) - a^2(b+c)^2 + 2abc(b+c) - 2b^2c^2) = 0.$

We have:

$$\left[\begin{array}{l} b - c = 0 \\ a^3(b+c) - a^2(b+c)^2 + 2abc(b+c) - 2b^2c^2 = 0 \end{array}\right.$$

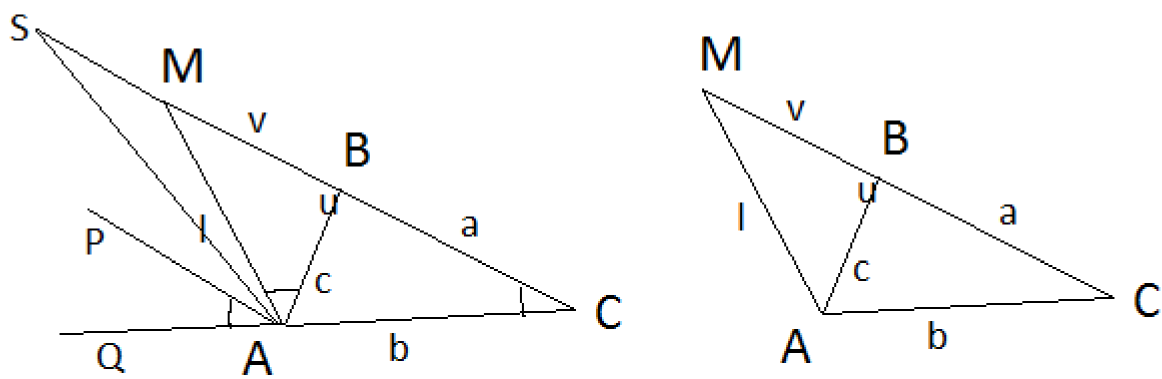
Now we are proving that the second is equal to zero only if $b = 0$ and $c = 0$.

$$a^3(b+c)-a^2(b+c)^2+2abc(b+c)-2b^2c^2 = 0 \quad \Leftrightarrow \quad a^3b+a^3c+2abc^2+2ab^2c = a^2b^2+2a^2bc+a^2c^2+2b^2c^2 \quad \Leftrightarrow$$

$$a^2b(a-b)+a^2c(a-c)+2bc(ab+ac-a^2-bc) = 0 \quad \Leftrightarrow \quad a^2b(a-b)+a^2c(a-c)+2bc(b-a)(a-c) = 0.$$

Since $a < c$ and $a < b$ $\quad a^2b(a-b)+a^2c(a-c)+2bc(b-a)(a-c) = 0 \quad \Leftrightarrow \quad b = c = 0.$

4 Is it true that two exsymmedians of a triangle are equal if and only if the triangle is isosceles?



Proposition 4.1.

Let l be a exsymmedian. $\frac{v}{u} = \frac{c^2}{b^2}$.

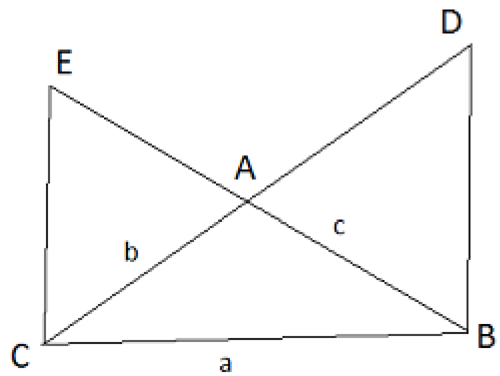
Proof. Let AS be a external bisector and AP be an exmedian. $\angle MAB = \angle PAQ = \angle BCA \quad (PA \| BC)$. Triangles MBA and MAC are similar so $\frac{v}{c} = \frac{l}{b}$ and $\frac{l}{c} = \frac{u}{b}$. So $v = \frac{c*l}{b}$ and $u = \frac{l*b}{c}$. So $\frac{u}{v} = \frac{b^2}{c^2}$. □

Now we are finding the length of a exsymmedian.

According to the Stuart's theorem: $c^2 = l^2\frac{a}{a+v} + b^2\frac{v}{a+v} - av = l^2\frac{a(b^2-c^2)}{ab^2} + \frac{b^2ac^2}{b^2-c^2} * \frac{b^2-c^2}{ab^2} - \frac{a^2c^2}{b^2-c^2}$.

$l^2\frac{b^2-c^2}{b^2} = \frac{a^2c^2}{b^2-c^2}$. So $l^2 = \frac{a^2b^2c^2}{(b^2-c^2)^2}$. So $l = \frac{abc}{b^2-c^2}$.

Consider a triangle ABC with exsymmedians CE and BD.



$$CE = BD \quad \Leftrightarrow \quad \frac{abc}{a^2 - b^2} = \frac{abc}{a^2 - c^2} \quad \Leftrightarrow \quad a^2 - b^2 = a^2 - c^2 \quad \Leftrightarrow \quad b = c.$$

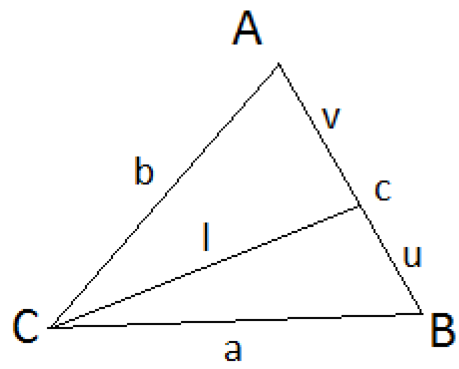
5 Check that the internal bisectors and the symmedians are respectively internal 1-lines and 2-lines of the triangle. Also, the external bisectors and the exsymmedians are respectively external 1-lines and 2-lines of the triangle.

It is a well-known fact that internal and external bisectors are 1-lines of the triangle and they can be find in literature. The proofs that symmedians and exsymmedians are given in the propositions 2.1 and 4.1.

6 Is it true that two internal n-lines of a triangle are equal if and only if the triangle is isosceles?

Now we are finding the length of internal n-line.

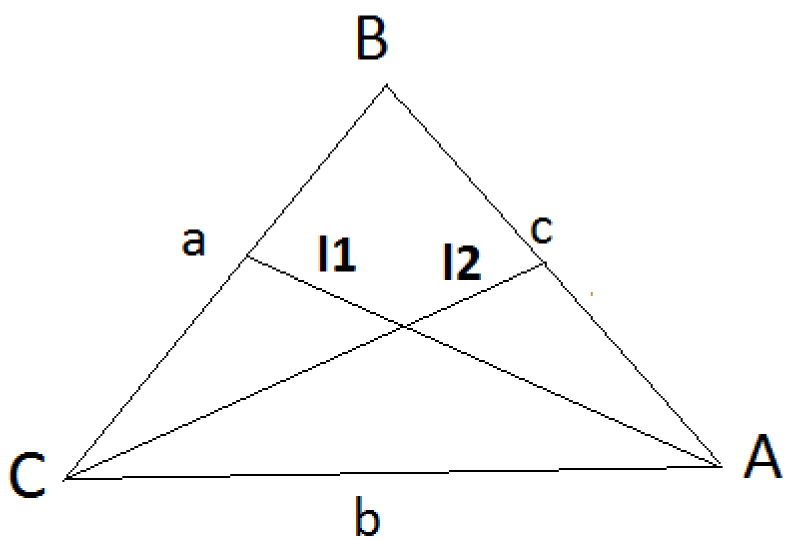
Consider triangle ABC with the internal n-line l.



$\frac{u}{v} = \frac{a^n}{b^n}$ so $\frac{u}{a^n} = \frac{v}{b^n}$. By property of proportion $\frac{v}{b^n} = \frac{u}{a^n} = \frac{u+v}{a^n+b^n}$. So $u = \frac{a^n c}{a^n+b^n}$ and $v = \frac{b^n c}{a^n+b^n}$.

According to the Stuart's theorem $l^2 = b^2\frac{u}{c} + a^2\frac{v}{c} - uv = b^2\frac{a^n}{a^n+b^n} + a^2\frac{b^n}{a^n+b^n} - \frac{a^n b^n c^2}{(a^n+b^n)^2}$

Consider triangle ABC with the internal n-lines l_1 and l_2.

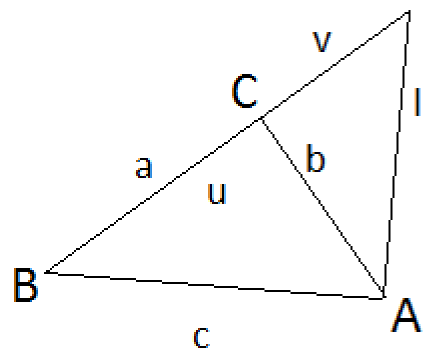


$$l_1 = l_2 \quad \Leftrightarrow \quad b^2\frac{a^n}{a^n+b^n} + a^2\frac{b^n}{a^n+b^n} - \frac{a^n b^n c^2}{(a^n+b^n)^2} = b^2\frac{c^n}{b^n+c^n} + c^2\frac{b^n}{b^n+c^n} - \frac{a^2 b^n c^n}{(b^n+c^n)^2}$$

So we have to prove that there exists just one solution of this equation $(a = c)$.

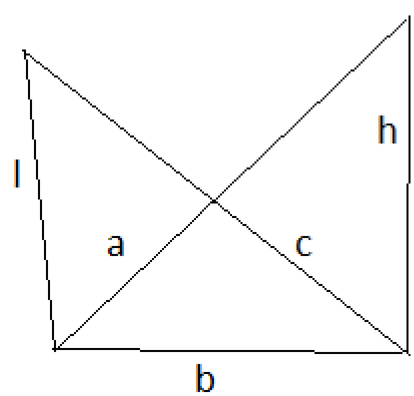
7 Is it true two external n-lines of a triangle are equal if and only if the triangle is isosceles?

Now we are finding the length of internal n-line.



$\frac{u}{v} = \frac{b^n}{c^n}$ so $\frac{u}{b^n} = \frac{v}{c^n}$. By property of proportion $\frac{u}{b^n} = \frac{v}{c^n} = \frac{u-v}{b^n-c^n} = \frac{a}{b^n-c^n}$. So $u = \frac{ab^n}{b^n-c^n}$ and $v = \frac{ac^n}{b^n-c^n}$.

By Stuart's theorem $c^2 = b^2\frac{v}{u} + l^2\frac{a}{u} - av = b^2\frac{c^n}{b^n} + l^2\frac{b^n-c^n}{b^n} - \frac{a^2c^n}{b^n-c^n}$. So $l^2 = \frac{c^2b^n}{b^n-c^n} + \frac{a^2b^nc^n}{(b^n-c^n)^2} - \frac{b^2c^n}{b^n-c^n}$.



$$l = h \quad \Leftrightarrow \quad \frac{c^2b^n}{b^n-c^n} + \frac{a^2b^nc^n}{(b^n-c^n)^2} - \frac{b^2c^n}{b^n-c^n} = \frac{a^2b^n}{b^n-a^n} + \frac{a^nb^nc^2}{(b^n-a^n)^2} - \frac{b^2a^n}{b^n-a^n}.$$

So we have to prove that there exists just one solution of this equation $(a = c)$.

References:

1. http://en.wikipedia.org/wiki/Bisection

2. http://www.cut-the-knot.org/Curriculum/Geometry/ExternalAngleBisector.shtml

3. http://en.wikipedia.org/wiki/Stewart's$_t$*heorem*