**Team: Russia 2**

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# Problem №10
# Densities of Natural Subsets

## *Question 1*

1) $\lim\limits_{n\longrightarrow\infty} \frac{\#(\text{ E}\cap[1,n])}{n}$

Obviously, the limit is equal to 0 for E - end. So, it makes sense and is therefore well defined.

2) $\lim\limits_{x\longrightarrow 1-}$ (1 - x) $\sum$x$^n$, when n $\in$ E

The series converges in the interval (0, 1), and hence the limit is meaningful and accurate value. So it is well defined.

3) $\lim\limits_{x\longrightarrow 1+}$(x - 1) $\sum\frac{1}{n^x}$, when n $\in$ E

If x$\in$ (0, n) the series converges for any $n$, and hence is well defined.

## *Question 2*

1) $\lim\limits_{n\longrightarrow\infty} \frac{\#(\text{ E}\cap[1,n])}{n} = 0$

### *Proof*

Indeed, we take E - a finite set.
Then the problem reduces to the calculation $\frac{\#\text{E}}{n}$, and it is obvious that $\#E = m$, m $\in \mathbb{N}$. n $\longrightarrow \infty \Longrightarrow n \longrightarrow 0$.

2) $\lim\limits_{x\longrightarrow 1-}$(1 - x) $\sum$x$^n$= 0, when n $\in$ E

### *Proof.*

Take a finite number of E. Then a finite number of elements, and thus equal to the number. And then the density is equal to 0.

3) $\lim\limits_{x\longrightarrow 1+}$(x - 1) $\sum\frac{1}{n^x}$= 0, when n $\in$ E
The proof is analogous to the second point. A series has a finite number of terms, and therefore - is convergent. And then the density is equal to 0.

## *Question 3*

1) $\lim\limits_{n\longrightarrow\infty} \frac{\#(\text{ E}\cap[1,n])}{n} = \frac{1}{d}$, when E = $\Big\{a + nd| \quad n, d\in \mathbb{N}\}$
The proof is obvious as $\mu_1$(E) can be thought of as $\frac{1}{\text{d}}$*$\mu_1(\mathbb{N}) \Longrightarrow \mu_1$(E) $= \frac{1}{\text{d}}$

## *Question 4*

1) $\lim\limits_{n \longrightarrow \infty} \frac{\#\ (\ \mathrm{E} \cap [1,n])}{n} = 0$, when E = $\Big\{[x^n]| \quad n \in \mathbb{N}\}$
Obviously, this is a geometric progression.
This can be proved in two ways:

### *First proof*

Take $a_n$ of E, and a with a radius r = min ($|a_{n-1}a_n|$, $|a_n a_{n+1}|$).

Starting from some point in this nearby will be n natural numbers (and their number tends to $\infty$ ), And only one number exponentially. It is obvious that at infinity, the ratio of $\frac{\#E}{n} \longrightarrow 0 \Longrightarrow \mu_1(\mathrm{E}) = 0$.

### *Second proof*

1) $\lim\limits_{n \longrightarrow \infty} \frac{\#\ (\ \mathrm{E} \cap [1,n])}{n} = \lim\limits_{n \longrightarrow \infty} \frac{\#E * \#\mathbb{N}}{\#\mathbb{N} * n}$.

But $\frac{\#\mathbb{N}}{n} \longrightarrow 1$, but $\frac{\#\mathrm{E}}{\#\mathbb{N}} \longrightarrow 0$ (this is obvious, since E - geometric progression).

Finally, $\mu_1(\mathrm{E}) = 0$.

3) $\lim\limits_{x \longrightarrow 1+}$ (x - 1) $\sum \frac{1}{n^x} = 0$ . It is clear that to prove this it suffices to prove the convergence $\sum \frac{1}{d^{nx}}$, and this series converges by Theorem of Leibniz.

## *Question 5*

The problem in the proof $\mu_1(\mathrm{A}) + \mu_1(\mathrm{B}) = \mu_1(\mathrm{A} \cup \mathrm{B}) + \mu_1(\mathrm{A} \cap \mathrm{B})$ or refute this assertion.

Let $\mathrm{A} \cap \mathrm{B} = \emptyset$.

That means we need only verify that, $\mu_1(\mathrm{A}) + \mu_1(\mathrm{B}) = \mu_1(\mathrm{A} \cup \mathrm{B})$.

## *Question 7*

$\mu_{2,3}(\mathrm{A}) + \mu_{2,3}(\mathrm{B}) = \mu_{2,3}(\mathrm{A} \cup \mathrm{B}) + \mu_{2,3}(\mathrm{A} \cap \mathrm{B})$

### *Proof*

Obviously, for $\mathrm{A} \cap \mathrm{B} = \emptyset$ this is done. Indeed, fix A and B, $\mathrm{A} \cap \mathrm{B} = \emptyset$.

$\mu_{2,3}(\mathrm{A} \cap \mathrm{B}) = 0$, but $\lim\limits_{x \longrightarrow 1-}$ (1 - x) $\sum \mathrm{x}^{\mathrm{n}}$(n $\in \mathrm{A} \cup \mathrm{B}$) = $\lim\limits_{x \longrightarrow 1-}$(1 - x)$\sum \mathrm{x}^n$(n $\in$A) + $\lim\limits_{x \longrightarrow 1-}$(1 - x) $\sum \mathrm{x}^{\mathrm{n}}$(n$\in$B) - since they are the exact limits.

This means that $\mu_2(\mathrm{A} \cup \mathrm{B}) = \mu_2(\mathrm{A}) + \mu_2(\mathrm{B})$. For the third density is similar.

Now we prove the general case. Fix A and B - arbitrary sets. Then we take X, Y, Z such as:

$X = A \setminus B$

$Y = A \cap B$

$Z = B \setminus A$

It is obvious X, Y, Z do not intersect.

Then $\mu_{2,3}(A \cup B) + \mu_{2,3}(A \cap B) = \mu_{2,3}(X \cup Y \cup Z) + \mu_{2,3}(Y) = \mu_{2,3}(X) + \mu_{2,3}(Y) + \mu_{2,3}(Z) + \mu_{2,3}(Y)$ by the above (since X, Y, Z do not intersect).

But this in turn equals $\mu_{2,3}(A) + \mu_{2,3}(B) \Longrightarrow \mu_{2,3}(A) + \mu_{2,3}(B) = \mu_{2,3}(A \cup B) + \mu_{2,3}(A \cap B)$.