Team: Russia 2

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Problem №1
Generalizing Perfectness

Abstract

In this paper we consider all suggested functions and find all positive integers which are perfect for them. For functions $f(n) = \tau(n) + k$ we find perfect numbers for $k = -1, 0$ and proved that there are no $\tau(n) + k$ - perfect numbers for $k > 1$. Also we generalized the third question for $f(n) = n + m$ and the sixth for $f(n) = \binom{m}{n}$. Furthermore we consider f - amicable numbers for some various arithmetic functions f. Considering the case when $f(n)$ is the nth member of geometric progression results in solving the equation $2q^n = \sum\limits_{d|n} q^d$. That task looks rather interesting but unfortunately we have not found any ways to solve it so we would be very grateful for those who can give an elegant solution for that case.

List of used theorems

1) $\tau(p_1^{\alpha_1}p_2^{\alpha_2}...p_s^{\alpha_s}) = (\alpha_1+1)(\alpha_2+1)...(\alpha_s+1)$

2) $\sigma(p_1^{\alpha_1}p_2^{\alpha_2}...p_s^{\alpha_s}) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdot ... \cdot \frac{p_s^{\alpha_s+1}-1}{p_s-1}$

3) $\sum\limits_{d|n}\varphi(d) = n$

4) φ - multiplicative function and $\varphi(p^\alpha) = p^{\alpha-1}(p-1)$

5) for $m > 1$ $\varphi(m) = m\prod\limits_{p|m}(1-\frac{1}{p})$

Where:

- $\tau(n)$ - the number of positive divisors of n
- $\sigma(n)$ - the sum of all positive divisors of n
- $\varphi(n)$ - the number of positive integers less than or equal to n that are relatively prime to n

All this theorems with proofs can be found in A. A. Buchstab "Number theory", Moscow, 1966

Solution of the problem

1st Question

To begin with we give the equivalent definiton for n $(n \neq 1)$ to be f - perfect:

$$2f(n) = \sum_{d|n} f(d)$$

Statement 1. n is f - perfect for $f(n) = \tau(n) \Longleftrightarrow n = p^2$ where p is prime.

$$\tau(n) = \sum_{d|n} 1$$

As 1 and d is a divisors of any number d (except the case $d = 1$) we get:

$$\sum_{\substack{d|n \\ 1 \le d \le n-1}} 1 \ge \sum_{\substack{d|n \\ 1 \le d \le n-1}} 2 - 2$$

Therefore to get equality $\tau(n) = \sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} \tau(d)$ there should be only one divisor d $(d \neq n)$ with 2 own divisors.
Obviously it is possible only if $n = p^2$.
It is easy to check:

$$3 = \tau(p^2) = \sum_{\substack{d|p^2 \\ 1 \le d \le p^2-1}} \tau(d) = 1 + 2 = 3$$

Statement 2. n is f - perfect for $f(n) = \tau(n) - 1 \iff n = p^3$.

$$\sum_{\substack{d|n \\ 1 \le d \le n-1}} 1 = \tau(n) - 1 = \sum_{\substack{d|n \\ 1 \le d \le n-1}} (\tau(d) - 1)$$

Every divisor of n (except 1) has at least 2 own divisors. Therefore to get $+1$ for each divisor there should be exactly one divisor of n with 3 own divisors.

It is possible only in case:

$$n = p^3$$

$$3 = \tau(p^3) - 1 = (\tau(1) - 1) + (\tau(p) - 1) + (\tau(p^2) - 1) = 3$$

Lemma 1. For any $\tau(n) + k$ - perfect n ($k > 0$) we have the bound: $\tau(n) \le \frac{2k+1}{k}$.

$$2\tau(n) + 2k = \sum_{d|n} (\tau(d) + k) = \sum_{d|n} \tau(d) + k\tau(n) \ge 2\tau(n) - 1 + k\tau(n)$$

$$\tau(n) \le \frac{2k+1}{k}$$

For the case $k < 0$ we can get lower bound for the amount of the divisors of n:

$$\tau(n) \ge \frac{2k+1}{k}$$

For $k = 1$ we get $\tau(n) \le 3$

$\bullet \tau(n) = 3 \Leftrightarrow n = p^2$
$4 = \tau(p^2) + 1 \ne \sum_{\substack{d|p^2 \\ 1 \le d \le p^2-1}} (\tau(d) + 1) = 5$
$\bullet \tau(n) = 2 \Leftrightarrow n = p$
$3 = \tau(p) + 1 \ne 2$
$\bullet \tau(n) = 1 \Leftrightarrow n = 1$ (that case has no sense because of bounds for divisor: $1 \le d \le n - 1$)
Thus, there are no $\tau(n) + 1$ - perfect numbers.

Statement 3. There are no $\tau(n) + k$ - perfect numbers for $k > 1$.

$$\tau(n) \le \frac{2k+1}{k} = 2 + \frac{1}{k}$$

Therefore for $f(n) = \tau(n) + k$ the only perfect numbers are primes.

$$k + 2 = \tau(p) + k \ne \sum_{\substack{d|p \\ 1 \le d \le p-1}} (\tau(d) + k) = k + 1$$

2nd Question

$\varphi(n) = \sum_{\substack{d|n \\ 1 \le d \le n-1}} \varphi(d) \Leftrightarrow 2\varphi(n) = n$ (according to the *Theorem 3*)

$$\varphi(n) = \frac{n}{2}$$

Statement 4. n is φ - perfect$\Longleftrightarrow n = 2^k$.

Let $n = 2^k p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}$ $(p_1 \neq p_2 \neq ... \neq p_s \neq 2)$
According to the *Theorems 4* and *5* we get:

$$\varphi(n) = 2^{k-1}\varphi(p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) = 2^{k-1}\frac{n}{2^k}(1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_s})$$

Therefore $\varphi(n) = \frac{n}{2} \Leftrightarrow (1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_s}) = 1$ what is impossible.
It is easy to check:

$$\varphi(2^n) = 2^{n-1} = \frac{2^n}{2}$$

3rd Question

Statement 5. If k is a natural number such that $p = 2^{k+1} + \frac{2mk}{l} - 1$ is a prime, then $n = 2^k(2^{k+1} + \frac{2mk}{l} - 1)$ is f - perfect for $f(n) = ln + m$.

$$n = 2^k p$$

$$2^{k+1}lp + 2m = l\sum_{d|n} d + m\tau(n)$$

According to the *Theorems 1* and *2* we get:

$$2^{k+1}lp + 2m = 2^{k+1}lp + 2^{k+1}l - pl - l + 2mk + 2m$$

$$p = 2^{k+1} + 2mk - 1$$

4th Question

Lemma 2. $\prod_{d|n} d = n^{\frac{\tau(n)}{2}}$

Note that if d is a divisor of n then $\frac{n}{d}$ is also divisor of n.
Therefore $\prod_{d|n}(d \cdot \frac{n}{d})$ is a square of required product and $\prod_{d|n}(d \cdot \frac{n}{d}) = n^{\tau(n)}$.
From it follows that $\prod_{d|n} d = n^{\frac{\tau(n)}{2}}$.

Statement 6. n is ln - perfect $\Longleftrightarrow n = p^3 \bigvee n = p_1 p_2$.

$$2ln(n) = \sum_{d|n} ln(d) = ln(n^{\frac{\tau(n)}{2}})$$

$$n^2 = n^{\frac{\tau(n)}{2}}$$

$$\tau(n) = 4$$

Obviously it is possible only in cases $n = p^3$ and $n = p_1 p_2$
Lets check:

$$ln(p^6) = ln(1 \cdot p \cdot p^2 \cdot p^3)$$

$$ln(p_1^2 p_2^2) = ln(1 \cdot p_1 \cdot p_2 \cdot p_1 p_2)$$

5th Question

For function $f(n) = (-1)^n$ we consider 2 cases:
. n ***is odd***

Condition for n to be f - perfect:

$$-1 = \sum_{\substack{d|n \\ 1 \le d \le n-1}} (-1)$$

Only prime n can satisfy this condition.

. n ***is even***

Condition for n to be f - perfect:

$$2 = \sum_{d|n} (-1)^d$$

That means there should be exactly 2 more even divisors than odd.

$$n = 2(2^{k-1} p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s})$$

Number of even divisors according to the *Theorem 1*:

$$k(\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_s + 1)$$

Number of odd divisors:

$$(\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_s + 1)$$

So we get:

$$k \prod_{i=1}^{s} (\alpha_i + 1) - 2 = \prod_{i=1}^{s} (\alpha_i + 1)$$

$$\prod_{i=1}^{s} (\alpha_i + 1) = \frac{2}{k-1} \ (k \neq 1)$$

As $\prod_{i=1}^{s} (\alpha_i + 1)$ and k are positive integers there are only 2 possible cases:

. $k = 2$:

$$n = 4p$$

. $k = 3$:

$$n = 4$$

That means the only f - perfect numbers for $f(n) = (-1)^n$ are $n = p \bigvee n = 4 \bigvee n = 4p$

6th Question

Statement 7. $n = m-1$ is f - perfect for $f(n) = \binom{m}{n}$ if $m-1$ is a prime.

Lets prove that:

$$\binom{m}{m-1} = \sum_{\substack{d|m-1 \\ 1\le d\le m-2}} \binom{m}{d}$$

It is equall:

$$\frac{m!}{(m-1)!1!} = \sum_{\substack{d|m-1 \\ 1\le d\le m-2}} \frac{m!}{d!(m-d)!}$$

But as $m-1$ is a prime number we get the identity:

$$\frac{m!}{(m-1)!1!} = \frac{m!}{1!(m-1)!}$$

Statement 8. $f(n) = \binom{m}{n}$ is a decreasing function when $\frac{m-1}{2} < n < m$ and increasing when $1 < n < \frac{m-1}{2}$

Consider the fraction $\frac{f(n+1)}{f(n)}$:

$$\frac{(m)!(n)!(m-n)!}{(n+1)!(m-n-1)!m!} = \frac{m-n}{n+1}$$

$$\frac{m-n}{n+1} < 1 \Leftrightarrow \frac{m-1}{2} < n$$

$$\frac{m-n}{n+1} > 1 \Leftrightarrow n < \frac{m-1}{2}$$

Statement 9. There are no f - perfect numbers for $f(n) = \binom{m}{n}$ which are less than $m-1$ if $m-1$ is a prime number.

Note that if n is a f - perfect number then $m-n$ also should be f - perfect.
While $\frac{m-1}{2} < n < m-1$ any divisor of n d is less than $\frac{m-1}{2}$. That means for such n f - is decreasing function and for such d is increasing.
Sum of increasing functions is also increasing.
Therefore the equation:

$$\binom{m}{n} = \sum_{\substack{d|n \\ 1\le d\le n-1}} \binom{m}{d}$$

Can have only one solution which we have already found.
But as there are no solutions for $\frac{m-1}{2} < n < m-1$ there are also no solutions for $1 < n < \frac{m-1}{2}$.

Conclusion. If $m-1$ is a prime then $n = m-1$ is the only f - perfect number for $f(n) = \binom{m}{n}$.

For example when $m = 2012$ the only f - perfect number is 2011.

7th Question

Lemma 3. For any positive decreasing arithmetic function $f : \mathbb{N} \to \mathbb{R}$ there are no f - perfect numbers.

Every divisor of n (excluding n) is less than n. Therefore $f(d) > f(n)$ and as the function is positive we get:

$$f(n) < \sum_{\substack{d|n \\ 1 \leq d \leq n-1}} f(d)$$

According to *Lemma 3* there is a whole set of functions without f - perfect numbers. For example $f(n) = \frac{a}{b^n + c}$ when $b > 1$, $a > 0$, $c > 0$.

Statement 10. For function $f(n) = klog_a(n)$ the only f - perfect numbers are such n with $4k$ divisors.

Note that the base of the logarithm does not matter and this task is equall to solving the question for $f(n) = kln(n)$.

According to *Lemma 2* we get:

$$2kln(n) = ln(n^{\frac{\tau(n)}{2}})$$

$$ln(n^{2k}) = ln(n^{\frac{\tau(n)}{2}})$$

$$\tau(n) = 4k$$

8th Question

Definition. We say that m and n f - amicable if $f(n) = \sum_{\substack{d|m \\ 1 \leq d \leq m-1}} f(d)$ and $f(m) = \sum_{\substack{d|n \\ 1 \leq d \leq n-1}} f(d)$.

Thus, m and n f - amicable for $f(k) = k$ if and only if m and n are amicable.
For function $f(k) = c$ (c is some constant) only $m = p_1$ and $n = p_2$ can be f - amicable.

Henceforth we denote f - amicable m and n as a pair (m, n).
Note that if (m, n) is amicable then (n, m) also should be f - amicable.

Statement 11. (m, n) is f - amicable for $f(k) = ln(k) \Longleftrightarrow (m, n) = (p_1 p_2^7), (p_1, p_2^3 p_3), (p_1, p_2 p_3 p_4), (p_1 p_2, p_3 p_4), (p_1^3, p_2 p_3), (p_1^3$ $(p_1 p_2, p_3^3)$

Proof is the similar to the *Statement 3.* We get:

$$n^2 = m^{\frac{\tau(m)}{2}}$$

$$m^2 = n^{\frac{\tau(n)}{2}}$$

Extend n from the second equation and substitute it to the first:

$$m^{\frac{8}{\tau(n)}} = m^{\frac{\tau(m)}{2}}$$

$$\tau(m)\tau(n) = 8$$

According to the fact τ is an integer function and the *Theorem 1* we get all these pairs.