Problem 6: Recurrent sequences

Russia #1, LCME, Saint-Petersburg

Properties of sequence u_n:

$$u_{n+1} = \frac{u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2}{n}$$

Statement 1 $u_{n+1} = \frac{u_n^2 + (n-1)u_n}{n},\ n \geq 3.$

Proof Let $\sigma_n = \sum\limits_{i=1}^{n} u_i^2$, so $u_{n+1} = \frac{\sigma_n}{n}$ *and* $\sigma_{n+1} = \sigma_n + \frac{\sigma_n}{n}^2$. Then, obviously

$$u_{n+1} = \frac{\sigma_n}{n} = \frac{\sigma_{n-1} + \frac{\sigma_{n-1}}{n-1}^2}{n} = \frac{u_n^2 + (n-1)u_n}{n}$$

Theorem 1 *The third element of sequence defines its finite monotonity:*
if $u_3 > 1$, *then* u_n *increases.*
if $u_3 = 1$, *then* u_n *is equal to 1.*
if $u_3 < 1$, *then* u_n *decreases.*

Proof Let's study $u_{n+1} - u_n$. According to statement 1:

$$u_{n+1} - u_n = \frac{u_n^2}{n} + \frac{n-1}{n}u_n - u_n = u_n\frac{u_n + n - 1 - n}{n} = \frac{u_n}{n}(u_n - 1).$$

As $\frac{u_n}{n}$ is always positive, the sign of the difference depends only on the sign of $(u_n - 1)$. Theorem statement obviously follows thence. Quod erat demonstrandum.

Theorem 2 *The third element of sequence defines its limit:*
if $u_3 > 1$, *then* $u_n \longrightarrow \infty$.
if $u_3 < 1$, *then* $u_n \longrightarrow 0$.

Proof We are going to provide proof only for the case when $u_3 < c < 1$. The other case is proved analogously.
It is obvious that there exists $p > 1$ such that $cp < 1$. We are going to proof (using mathematical induction) that for any $k \in \mathbb{N}$ s.f.s.n. $u_n < \frac{(cp)^{2^k}}{p}$.

1. Basis

$$u_{n+1} = \frac{u_1^2 + u_2^2 + \cdots + u_n^2}{n} < \frac{u_1^2 + u_2^2}{n} + \frac{n-2}{n}c^2.$$

$\frac{n-2}{n}c^2 < c^2$ and s.f.s.n. $\frac{u_1^2+u_2^2}{n} < c^2(p-1)$. Hence

$$\frac{u_1^2 + u_2^2}{n} + \frac{n-2}{n}c^2 < c^2p.$$

Quod erat demonstrandum.

2. Step
   Let u_k be less than $p^{2^i-1}c^{2^i}$ starting from n.

$$u_{k+1} = \frac{u_1^2 + u_2^2 + \cdots + u_n^2 + \cdots + u_k^2}{k} < \frac{u_1^2 + u_2^2}{k} + \sum_{i=1}^{s} \frac{const * p^{2^i-1}c^{2^i}}{k} + \frac{k-n}{k} p^{2^{s+1}-2}c^{2^{s+1}}.$$

   First two addends of sum converge to zero.
   So s.f.s.n. they are less than $c^{2^{s+1}}p^{2^{s+1}-2}(p-1)$. Hence

$$\frac{u_1^2 + u_2^2}{k} + \sum_{i=1}^{s} \frac{const * p^{2^i-1}c^{2^i}}{k} + \frac{k-n}{k} p^{2^{s+1}-2}c^{2^{s+1}} < p^{2^{s+1}-1}c^{2^{s+1}}.$$

   Quod erat demonstrandum.

Properties of sequence u_n:

$$u_{n+1} = \frac{u_1 * u_n + u_2 * u_{n-1} + \cdots + u_{n-1} * u_2 + u_n * u_1}{n}$$

Theorem 3 *if $u_2 > u_1 > 1$ then u_n increases. if $u_2 < u_1 < 1$ then u_n decreases.*

Proof

$$u_{n+1} = \frac{u_1 * u_n + u_2 * u_{n-1} + \cdots + u_{n-1} * u_2 + u_n * u_1}{n} =$$
$$\frac{(u_1 * u_n + u_2 * u_{n-1} + \cdots + u_{n-1} * u_2 + u_n * u_1) * (1 + \frac{1}{n})}{n * \frac{n+1}{n}} =$$
$$\frac{u_1 * u_n + u_2 * u_{n-1} + \cdots + u_{n-1} * u_2 + u_n * u_1 + u_{n+1}}{n+1}.$$

Consider defference $u_{n+2} - u_{n+1}$:

$$u_{n+2} - u_{n+1} = \frac{u_1(u_{n+1} - u_n) + u_2(u_n - u_{n-1}) + \cdots + u_n(u_2 - u_1) + u_{n+1}(u_1 - 1)}{n+1}.$$

All differences in brackets are simultaneously either positive or negative (altogether) hence it is obvious that sequence either increases or decreases according to the theorem statement.