Russia #1, LCME Saint-Petersburg

Problem 5: Stable Polygons

Abstract

We obtained some common results about the problem and made a conjection about the structure of stable polygons. This conjection is based on computer tests. According to the conjection we did 1,2 and 3 points of the problem.

Remark:

When we use this notation: P_n, we mean that $P_n = \{P_0, P_1, P_2, ..., P_{n-1}\}$ is a set of vertices of a regular n-gon.

Lemma 1. *Let $m \in \mathbb{N}$ and $m|n \Longrightarrow \forall A \subset P_n$, if $|A| = m$ and A is a regular m-gon, $A = \{P_k, P_{\frac{n}{m}+k}, ..., P_{\frac{n(m-1)}{m}+k}\}_{k=\overline{0,\frac{n}{m}-1}}$.*

Proof. It is clear that if $A \subset P_n$, $|A| = m$ and A is a regular m-gon, then $\forall P_i, P_j \in A$, where P_i, P_j are sequenced vertices of A, $j-i = \frac{n}{m}$. Then if P_l is a vertex of A with the lawest index, $A = \{P_l, P_{\frac{n}{m}+l}, ..., P_{\frac{n(m-1)}{m}+l}\}$ and $0 \leq l \leq \frac{n}{m} - 1$. $\square$

Lemma 2. *If $A \subset P_n$ and A is a regular m-gon $\Longrightarrow m|n$*

Lemma 3. *If $\forall A \subset P_n$, A is **not** regular m-gon $\Longleftrightarrow n$ is a prime number.*

Conjecture 4. *$A \subset P_n$ A - stable $\Longleftrightarrow A = \bigsqcup_{i \in J} B_i$, where $\forall i \in J$ B_i* is a regular and $\forall i, j \in J$, $i \neq j$ $B_i \cap B_j = \varnothing$.

1 First problem

When n is prime, find the number of stable subsets $A \subseteq P_n$ and describe them.

Theorem 5. *If n is prime and $A \subseteq P_n$, A - stable $\Longleftrightarrow$ $A = P_n$*

Proof. $\Longrightarrow$:

As n is prime, according to Lemma 3 and the Conjecture, P_n does not have stable subsets except itself. Then $\forall A \subseteq P_n$ if a A - stable $\Longrightarrow$ $A = P_n$.

$\Longleftarrow$:

$A = P_n$, then A - is regular. Then A is stable. $\square$

Corollary 6. *When n is prime P_n has only 1(one) stable subset - P_n.*

2 Second problem

The same problem when n is the product of two distinct prime numbers.

Theorem 7. *$\forall A, B \subset P_n : A, B$ are regular, $|A| = p$ and $|B| = q$, $p > q$ and q is not a divisior of p, $pq = n$ $\Longrightarrow A \cap B \neq \varnothing$*

Proof. Without loss of generality, we can claim that $A = \{P_0, P_q, ..., P_{q(p-1)}\}$ and $B = \{P_k, P_{p+k}, ..., P_{p(q-1)+k}\}$, where $k = \overline{0, q-1}$.

1. Let's prove that $\{pj\,(mod\,q) | j = \overline{0, q-1}\} = \{0, 1, 2, ..., q-1\}$:

To prove it Let $\alpha_1, \alpha_2 \in \{0, 1, 2, ..., q-1\}$ and $\alpha_1 < \alpha_2$. Then if $p\alpha_1\,(mod\,q) = p\alpha_2\,(mod\,q) \Longrightarrow p(\alpha_2 - \alpha_1) = ql$, where $l \in \mathbb{N}$, but q is not divisor of p and $(\alpha_2 - \alpha_1)$. Controversial. Then this set $\{pj\,(mod\,q) | j = \overline{0, q-1}\}$ consists of q different numbers from 0 to q-1.

2. As $\forall i = \overline{0, q-1}$ $qi = 0\,(mod\,q)$ and for $\forall k = \overline{0, q-1}$ $\exists j = \overline{0, q-1}$ such that $pj = 0\,(mod\,q)$, then $\{P_{qi}\} = \{P_{pj}\} \in A \cap B$. That is $A \cap B \neq \varnothing$. $\square$

Theorem 8. *If n=pq, where $p \neq q$ and p,q are primes, then for $A \subseteq P_n$ A - stable* $\Longleftrightarrow A = \bigsqcup_{i \in J_1} M_i$*, where* M_i is a regular p-gon or $A = \bigsqcup_{j \in J_2} N_j$*, where* N_j is a regular q-gon.

Proof. $\Longrightarrow$:

Let A be a stable subset of P_n. Then, according to the Conjecture, $A = \bigsqcup_{i \in J} B_i$, where $\forall i \in J$ B_i is a regular. That is, the only thing that we need prove is that A cannot consist of p-gons and q-gons at the same time. It is true because $n = pq$ and due to the Theorem 7 for any M - p-gon and for any N - q-gon, $M \cap N \neq \varnothing$.

$\Longleftarrow$:

$<$It is true because of the Conjecture$>$ □

Claim 9. Because of the Theorem 8, the number of stable subsets of P_n , S $= C_q^1 + C_q^2 + ... + C_q^q + C_p^1 + C_p^2 + ... + C_p^p - 1 = 2^q - 1 + 2^p - 1 - 1 = 2^q + 2^p - 3$

Corollary 10. *When n=pq, all the stable subsets of P_n are described by the Theorem 8 and their number is* $2^q + 2^p - 3$.

3 Third problem

The same problem when n is a power of a prime number.

Claim 11. As all the regular subsets of P_n can be obtained as a union of p-gons, if $n = p^k$, where p is prime and $k \in \mathbb{N}$, according to the Conjecture, all the stable subsets A of P_n are described: $A = \bigsqcup_{i \in J} B_i$,where B_i is a regular p-gon.

Claim 12. Because of the Claim 11, the number of stable subsets of P_n , S $= C_p^1 + C_p^2 + C_p^3 + ... + C_p^p = 2^p - 1$

Corollary 13. *When $n = p^k$, all the stable subsets of P_n are described by the Claim 11 and their number is* $2^p - 1$.