# Problem 4: **Isosceles Triangles**

Russia #1, Saint-Petersburg, Russia

**1.** Prove that two internal angle bisectors of triangle are equal if and only if the triangle is isosceles.

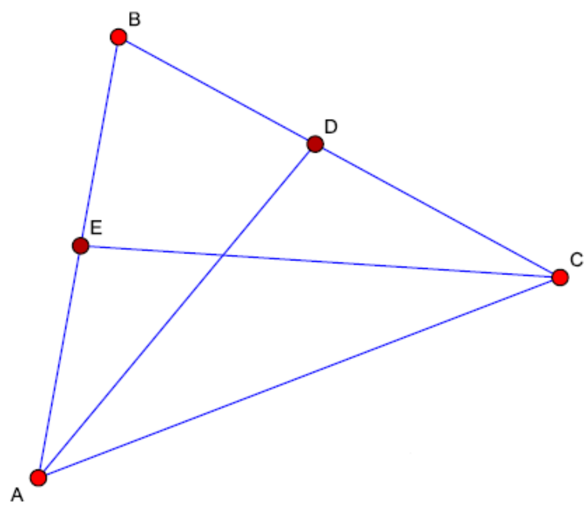


Figure 1: $\Delta ABC$ with two internal angle bisectors $AD = l_a$ and $CE = l_c$

Use the formula of length of internal angle bisector:

$$l_a = \frac{\sqrt{bc(a+b+c)(b+c-a)}}{b+c}$$

$$l_b = \frac{\sqrt{ac(a+b+c)(a+c-b)}}{a+c}$$

$$l_c = \frac{\sqrt{ab(a+b+c)(a+b-c)}}{a+b}$$

a) Assume that two internal angle bisectors are equal:

$$l_a = l_c \iff \frac{\sqrt{bc(a+b+c)(b+c-a)}}{b+c} = \frac{\sqrt{ab(a+b+c)(a+b-c)}}{a+b} \iff$$

$$\frac{c(b+c-a)}{(b+c)^2} = \frac{a(a+b-c)}{(a+b)^2} \iff c(b+c-a)(a+b)^2 = a(a+b-c)(b+c)^2 \iff$$

$$(c-a)(b(b+c)(a+b) + ac(a+2b+c)) = 0 \iff c-a=0 \iff a=c$$

b) Assume that the triangle $\Delta ABC$ is isosceles ($AB = BC$). Then triangles $\Delta ADC$ and $\Delta CEA$ ($\angle DAC = \angle ECA, \angle DCA = \angle EAC$ and $AC = AC$) are equal and $l_a = l_c$.

**2.** Prove that two symmedians of triangle are equal if and only if the triangle is isosceles.

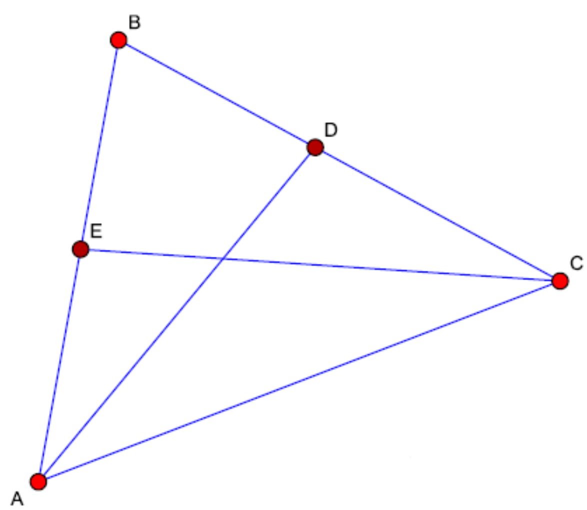


Figure 2: $\Delta ABC$ with two symmedians $AD = s_a$ and $CE = s_c$

$\cos\angle ABC = \frac{a^2+c^2-b^2}{2ac}$ and $BD = \frac{c^2}{b^2+c^2}a$ (5.b) then

$$s_a^2 = AB^2 + BD^2 - 2 \cdot AB \cdot BD \cos \angle ABC$$

$$s_a^2 = \frac{b^2c^2}{(b^2+c^2)^2}(2b^2+2c^2-a^2)$$

$$s_a = \frac{bc}{(b^2+c^2)}\sqrt{2b^2+2c^2-a^2}$$

Use the formula of length of symmedian:

$$s_a = \frac{bc}{b^2+c^2}\sqrt{2b^2+2c^2-a^2}$$

$$s_b = \frac{ac}{a^2+c^2}\sqrt{2a^2+2c^2-b^2}$$

$$s_c = \frac{ab}{a^2+b^2}\sqrt{2a^2+2b^2-c^2}$$

a) Assume that two symmedians are equal:

$$s_a = s_b \iff \frac{bc}{b^2+c^2}\sqrt{2b^2+2c^2-a^2} = \frac{ac}{a^2+c^2}\sqrt{2a^2+2c^2-b^2} \iff$$

$$\frac{b^2}{(b^2+c^2)^2}(2b^2+2c^2-a^2) = \frac{a^2}{(a^2+c^2)^2}(2a^2+2c^2-b^2) \iff$$

$$b^2(a^2+c^2)^2(2b^2+2c^2-a^2) = a^2(b^2+c^2)^2(2a^2+2c^2-b^2) \iff$$

$$a^2b^2(b^4-a^4) + 4a^2b^2c^2(b^2-a^2) + 2c^4(b^4-a^4) + 2c^6(b^2-a^2) = 0 \iff$$

$$(b-a)A = 0, \text{ there } A > 0 \iff b-a=0 \iff a=b$$

b) Assume that the triangle $\Delta ABC$ is isosceles ($AB = BC$). Then triangles $\Delta ADC$ and $\Delta CEA$ ($AE = DC$ (5.b), $\angle DCA = \angle EAC$ and $AC = AC$) are equal and $s_a = s_c$.

**4.** Is it true that two exsymmedians of triangle are equal if and only if the triangle is isosceles?

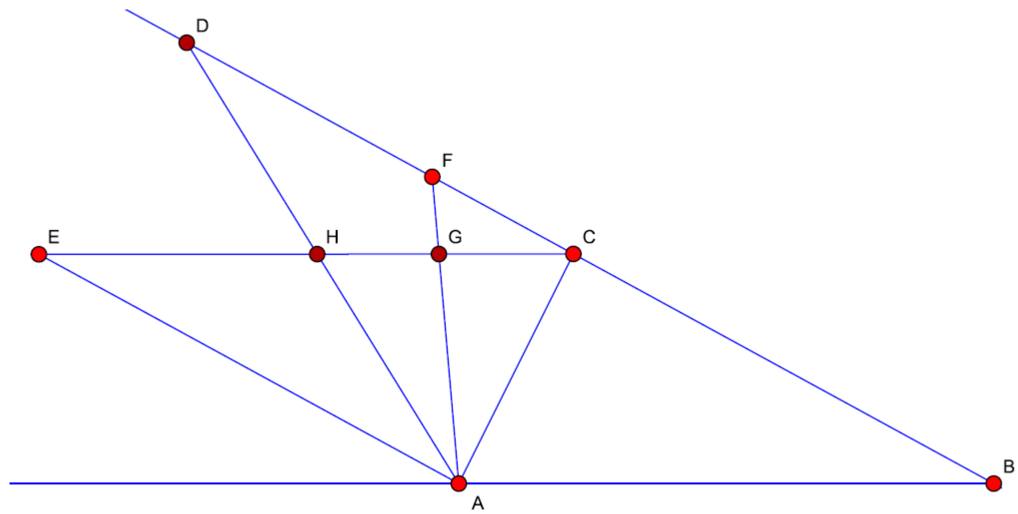


Figure 3: $\Delta ABC$ with exmedian $AE$, external angle bisector $AD$ and exsymmedian $AF$

Assume that $AB > AC$

$$\frac{AG}{BC} = \frac{GC}{AC} = \frac{AC}{AB} = \frac{b}{c}$$

$$\frac{AF}{FB} = \frac{FC}{AF} = \frac{AC}{AB} = \frac{b}{c}$$

$$\frac{FC}{FB} = \frac{FG}{FA} = \frac{GC}{AB} = \frac{b^2}{c^2}$$

then $AG = \frac{BC \cdot AC}{AB}$ and $GF = AG \cdot (1 + \frac{AC^2}{AB^2 - AC^2})$. Then $AF = AG + GF = \frac{AB \cdot BC \cdot CA}{AB^2 - AC^2} = \frac{abc}{c^2 - b^2}$

Use formula of length of exsymmedian

$$e_a = \frac{abs}{|c^2 - b^2|}$$

$$e_b = \frac{abs}{|a^2 - b^2|}$$

$$e_c = \frac{abs}{|a^2 - b^2|}$$

a) Assume that two exsymmedians are equal then

$$e_a = e_b \iff \frac{abc}{|b^2 - c^2|} = \frac{abc}{|a^2 - c^2|} \iff |b^2 - c^2| = |a^2 - c^2|$$

For example, $b = 5, c = 4, a = \sqrt{7}$, then $e_a = e_b$ and the triangular is not isosceles.

**5.** Check that the internal bisectors and the symmedians are respectively internal 1-lines and 2-lines of triangle. Also, the external bisectors and the exsymmedians are respectively external 1-lines and 2-lines of triangle.

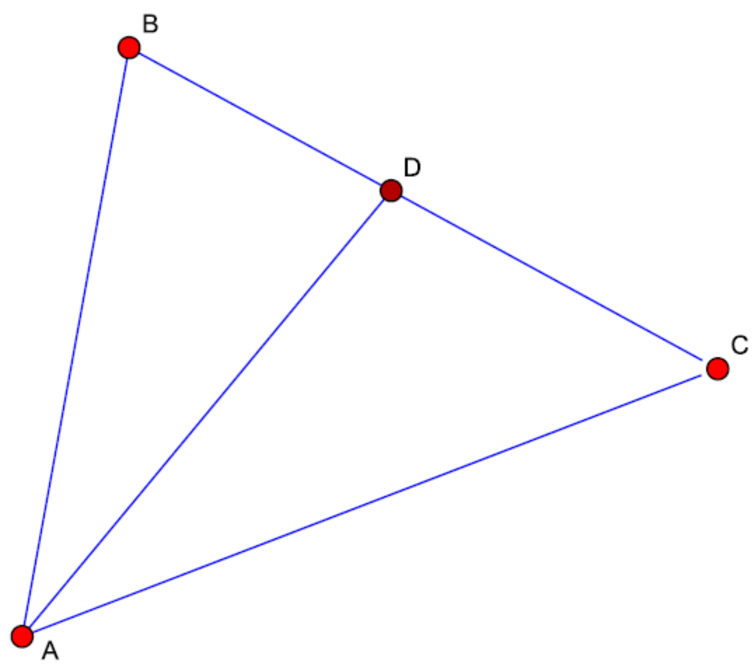


Figure 4: $\Delta ABC$ with internal angle bisector $AD$

a) Internal angle bisector (Figure 4): $\frac{BD}{DC} = \frac{S_{\Delta ABD}}{S_{\Delta ACD}} = \frac{AB \sin \angle BAD}{AC \sin \angle DAC}$. If $AD$ is internal angle bisector then

$\angle BAD = \angle DAC$ and $\frac{Bd}{DC} = \frac{AB}{AC} = \frac{c}{b}$

b) Symmedian (Figure 5): $BF = \frac{ac^2}{b^2+c^2}$, $FD = \frac{abc(b-c)}{b^2+c^2} \cdot \frac{1}{b+c}$, $DC = \frac{b}{b+c}a$, then

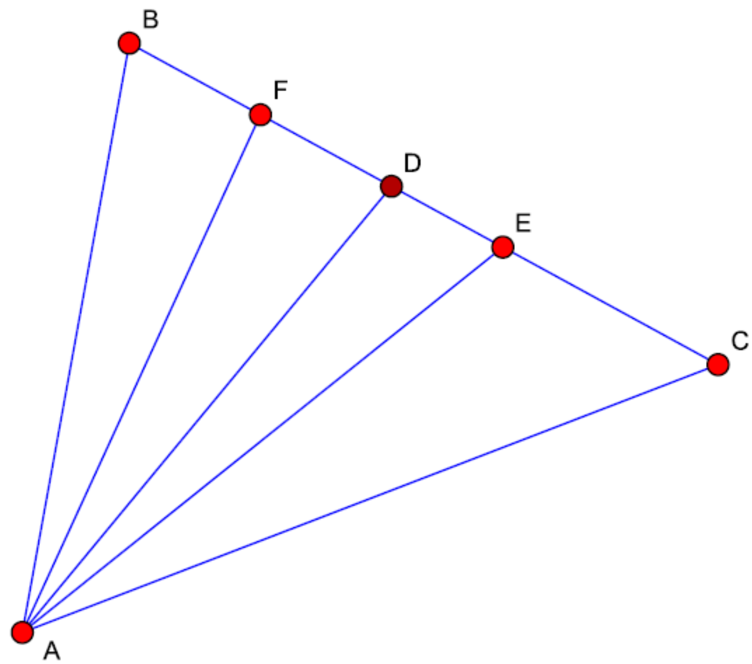


Figure 5: $\Delta ABC$ with internal angle bisector $AD$, median $AE$ and symmedian $AF$

$$\frac{BF}{FC} = \frac{BF}{FD + DC} = \frac{c^2}{b^2}$$

c) Exsymmedian (Figure 3): $\frac{FC}{FB} = \frac{b^2}{c^2}$ (4)

d) External angle bisector (Figure 3): $DF = AF$ ($\angle FDA = \angle FAD$) then $\frac{DC}{DB} = \frac{DF+FC}{DF+FC+CB} = \frac{b}{c}$