Problem 10: Densities of Natural Subsets

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Abstract

In this paper we were considering densities of natural subsets. In the first section we solved parts 1 and 2 of the problem. In the second section we created a theory generalizing those densities for some kinds of sequences and using these results we solved parts 3 and 4 of the problem. In the third section we solved part 6 and in the fourth section we proved a theorem that states we can build a set with any real density in segment $[0,1]$. This is related to part 5 of the problem.

1 Basic definitions

For any non-empty set $E \subset \mathbb{N}$ the following functionals are defined:

$$\mu_1(E) = \limsup_{n\to+\infty} \frac{\#\left(E \cap [1,n]\right)}{n},$$

$$\mu_2(E) = \limsup_{x\to 1^-} (1-x) \sum_{n\in E} x^n,$$

$$\mu_3(E) = \limsup_{x\to 1^+} (x-1) \sum_{n\in E} \frac{1}{n^x}.$$

Before investigating their properties, it is necessary to prove that these densities are well-defined. The definition of μ_1 can be rewritten as follows:

$$\mu_1(E) = \lim_{n\to+\infty} \sup_{k\geq n} \frac{\#\left(E \cap [1,k]\right)}{k}.$$

The supremums are monotonically decreasing, so to show existence of $\mu_1(E)$ for every $E \subset \mathbb{N}$ we need to prove that the supremums sequence is bounded. It true since

$$0 \leq \frac{\#\left(E \cap [1,n]\right)}{n} \leq 1$$

for every $n \in \mathbb{N}$.

We can rewrite μ_2 and μ_3 similarly:

$$\mu_2(E) = \lim_{\delta\to 0+} \sup_{x\in O_\delta^-(1)} (1-x) \sum_{n\in E} x^n,$$

$$\mu_3(E) = \lim_{\delta\to 0+} \sup_{x\in O_\delta^+(1)} (x-1) \sum_{n\in E} \frac{1}{n^x}.$$

As it was in the previous case, the supremums are monotonically decreasing(as functions), and locally bounded:

$$0 \le (1-x) \sum_{n \in E} x^n \le (1-x) \sum_{n \in \mathbb{N}} x^n = (1-x) \frac{x}{1-x} = x \le 1,$$

$$0 \le \sup_{x \in O_\delta^+(1)} (x-1) \sum_{n \in E} \frac{1}{n^x} \le \sup_{x \in O_\delta^+(1)} (x-1) \sum_{n \in \mathbb{N}} \frac{1}{n^x} \longrightarrow_{x \to 1+} 1$$

since $\lim\limits_{x \to 1+} (x-1) \sum\limits_{n \in \mathbb{N}} \frac{1}{n^x} = 1$.[1]

Now we can prove the first theorem.

Theorem 1. *$\mu_i(E) = 0$ for any finite set $E \subset \mathbb{N}$, $1 \le i \le 3$.*

Proof. Given a finite set $E \subset \mathbb{N}$, suppose that $\#E = k, k \in \mathbb{N}$. Then

$$\mu_1(E) = \limsup_{n \to +\infty} \frac{\#(E \cap [1,n])}{n} \le \limsup_{n \to +\infty} \frac{k}{n} = 0,$$

$$\mu_2(E) = \limsup_{x \to 1^-} (1-x) \sum_{n \in E} x^n \le \limsup_{x \to 1^-} (1-x) \cdot k = k \cdot 0 = 0,$$

$$\mu_3(E) = \limsup_{x \to 1^+} (x-1) \sum_{n \in E} \frac{1}{n^x} \le \limsup_{x \to 1^+} (x-1) \cdot k = k \cdot 0 = 0.$$

On the other hand, $\mu_i(E) \ge 0$, then $\mu_i(E) = 0$, $1 \le i \le 3$. □

2 Densities of sequences

In the previous section we proved that a finite set always has a zero density, so in further work we can assume that a set is infinite.

Any infinite natural subset can be represented as a monotonically increasing sequence, and it is possible to define those densities for sequences.

Definition 1. Given a sequence $\{a_n\}_{n \in \mathbb{N}}$ such that $\lim\limits_{n \to +\infty} a_n = +\infty$. We define 3 densities of a_n as follows:

$$\mu_1(a_n) = \limsup_{n \to +\infty} \frac{max\,\{k | a_k \le n\}}{n},$$

$$\mu_2(a_n) = \limsup_{x \to 1^-} (1-x) \sum_{n \in \mathbb{N}} x^{a_n},$$

$$\mu_3(a_n) = \limsup_{x \to 1^+} (x-1) \sum_{n \in \mathbb{N}} \frac{1}{a_n^x}.$$

According to the definition 1, a sequence doesn't need to contain only natural numbers and can be, for example, a sequence of positive real numbers. We will define a class of "densible" sequences and then will work with this class.

Definition 2. A sequence $\{a_n\}_{n \in \mathbb{N}}, a_n \in \mathbb{R}$ is called densible if and only if it is monotonically non-decreasing and tends to the positive infinity.

Lemma 1. *For any densible sequences a_n, b_n if $a_n \leq b_n$ for any $n \in \mathbb{N}$, then $\mu_i(a_n) \geq \mu_i(b_n)$.*

Proof. This lemma can be proved by using the following inequalities:

$$max\left\{k|a_k \leq n\right\} \geq max\left\{k|b_k \leq n\right\},$$

$$\sum_{n\in\mathbb{N}} x^{a_n} \geq \sum_{n\in\mathbb{N}} x^{b_n},$$

$$\sum_{n\in\mathbb{N}} \frac{1}{a_n^x} \geq \sum_{n\in\mathbb{N}} \frac{1}{b_n^x},$$

where x is as in the corresponding density. □

Lemma 2. *Given densible sequences a_n, b_n, c_n. If $a_n \leq b_n \leq c_n$ for any $n \in \mathbb{N}$ and $\mu_i(a_n) = \mu_i(c_n) = \lambda$ then $\mu_i(b_n) = \lambda$.*

Proof.

$$\mu_i(a_n) \geq \mu_i(b_n) \geq \mu_i(c_n),$$

$$\mu_i(b_n) = \lambda.$$

□

Lemma 3. *For any densible sequence a_n and for any real number $t > 0$ the equation $\mu_i(a_n + t) = \mu_i(a_n)$ holds.*

Proof. First we shall prove this lemma for $i = 2$.

$$\mu_2(a_n + t) = \limsup_{x\to 1^-}(1-x)\sum_{n\in\mathbb{N}} x^{a_n+t} =$$

$$= \limsup_{x\to 1^-}(1-x)x^t\sum_{n\in\mathbb{N}} x^{a_n} = \limsup_{x\to 1^-}(1-x)\sum_{n\in\mathbb{N}} x^{a_n} = \mu_2(a_n)$$

since x^t tends to 1 while $x \to 1$.

Second we will prove the lemma for $i = 3$. Let us remember that while $f(y) = y^x$ is convergent when $x > 1$ and then $(a+b)^x - a^x \leq (2^{x-1}-1)a^x + 2^{x-1}b^x$ where $a, b > 0$. Consider the following difference for $x > 1$:

$$(x-1)\sum_{n\in\mathbb{N}} \frac{1}{a_n^x} - (x-1)\sum_{n\in\mathbb{N}} \frac{1}{(a_n+t)^x} = (x-1)\sum_{n\in\mathbb{N}} \frac{(a_n+t)^x - a_n^x}{a_n^x(a_n+t)^x} \leq$$

$$\leq (2^{x-1}-1)(x-1)\sum_{n\in\mathbb{N}} \frac{1}{(a_n+t)^x} + 2^{x-1}t^x(x-1)\sum_{n\in\mathbb{N}} \frac{1}{(a_n+t)^x a_n^x} \leq$$

$$\leq (2^{x-1}-1)(x-1)\sum_{n\in\mathbb{N}} \frac{1}{a_n^x} + 2^{x-1}t^x(x-1)\sum_{n\in\mathbb{N}} \frac{1}{n^2}.$$

Both of the summed expressions tends to zero while $x \to 1-$, then

$$\limsup_{x\to 1^-}(1-x)\sum_{n\in\mathbb{N}} x^{a_n} = \limsup_{x\to 1^-}(1-x)\sum_{n\in\mathbb{N}} x^{a_n+t},$$

$$\mu_3(a_n) = \mu_3(a_n + t).$$

Now we will prove this lemma for $i = 1$. To do that, we will prove it for $t \in \mathbb{N}$ and then for any real $t > 0$.

$$\mu_1(a_n + t) = \limsup_{n\to+\infty} \frac{max\,\{k|a_k + t \le n\}}{n} = \limsup_{n\to+\infty} \frac{max\,\{k|a_k \le n - t\}}{n} =$$

$$= \limsup_{n\to+\infty} \frac{max\,\{k|a_k \le n - t\}}{n - t} \cdot \frac{n - t}{n} = \limsup_{n\to+\infty} \frac{max\,\{k|a_k \le n - t\}}{n - t} = \mu_1(a_n).$$

Since $[t] \le t \le [t] + 1$, the equality holds for any real $t > 0$.

□

Lemma 4. *Let $a_n = \lambda^n$, where $\lambda > 1$ is a real number. Then $\mu_i(a_n) = 0$ for any $1 \le i \le 3$.*

Proof.

$$\mu_1(\lambda^n) = \limsup_{n\to+\infty} \frac{max\,\{k|\lambda^k \le n\}}{n} \le \limsup_{n\to+\infty} \frac{\log_\lambda n}{n} = 0.$$

$$\mu_2(\lambda^n) = \limsup_{x\to 1^-} (1 - x) \sum_{n\in\mathbb{N}} x^{\lambda^n} = 0$$

$$\mu_3(\lambda^n) = \limsup_{x\to 1^+} (x - 1) \sum_{n\in\mathbb{N}} \frac{1}{\lambda^{nx}} = \limsup_{x\to 1^+} (x - 1) \sum_{n\in\mathbb{N}} \frac{1}{(\lambda^x)^n} =$$

$$= \limsup_{x\to 1^+} (x - 1) \frac{\frac{1}{\lambda^x}}{1 - \frac{1}{\lambda^x}} = 0.$$

□

Lemma 5. *Given a densible sequence a_n. $\mu_i(\alpha a_n) = \frac{1}{\alpha}\mu_i(a_n)$ for any $\alpha \in \mathbb{N}$.*

Theorem 2. *Let $E = \{a + nd \,|\, n \in \mathbb{N}_0\}$ for some $a, d \in \mathbb{N}$. Then $\mu_i(E) = \frac{1}{d}$, $1 \le i \le 3$.*

Proof. Denote by x_n the sequence corresponding to E or, in other words, $x_n = a + nd$, $n \in \mathbb{N}_0$. Then $\mu_i(x_n) = \mu_i(E)$. Let $y_n = x_n + (d - a)$. Since x_n is densible, y_n is densible. It follows from lemma 3 that $\mu_i(x_n) = \mu_i(y_n) = \mu_i((n + 1)d) = \frac{1}{d}\mu_i(n + 1) = \frac{1}{d}$. Thus $\mu_i(E) = \frac{1}{d}$. □

Theorem 3. *Let $E = \{[x^n] \,|\, n \in \mathbb{N}_0\}$ where $x \in \mathbb{R}$ and $x \ge 1$. Then $\mu_i(E) = 0$ for any $1 \le i \le 3$.*

Proof. Assume that $x > 1$, because the case $x = 1$ needs no proof. Consider $a_n = [x^n], n \in \mathbb{N}_0$, which is densible. It is enough to prove that $\mu_i(a_n) = 0$, since $0 \le \mu_i(E) \le \mu_i(a_n)$. We have

$$x^n - 1 \le a_n \le x^n.$$

According to lemmas 3 and 4, $\mu_i(x^n - 1) = \mu_i(x^n) = 0$. Using lemma 2, we get $\mu_i(a_n) = 0$. □

Theorem 4. *$\mu_i(F_n) = 0$, where F_n is a Fibonacci sequence and $1 \le i \le 3$.*

Proof. The proof is based on lemmas 4 and 5. □

3 Additivity

Theorem 5. *There exist sets* $A, B \subset \mathbb{N}$ *such that* $A \cap B = \emptyset$, $\mu_1(A) = \frac{1}{2}$, $\mu_1(B) = \frac{1}{4}$, $\mu_1(A \cap B) < \frac{1}{2} + \frac{1}{4}$.

Proof. Let $f_E(n) = \frac{\#(E\cap[1,n])}{n}$. We will build A, B in the way that $f_E(n)$ will change from $\frac{1}{2}$ to $\frac{1}{4}$ and back at A, and from $\frac{1}{8}$ to $\frac{1}{4}$ and back at B. For that, we will build 2 monotonically increasing sequences a_n, b_n $(a_{n+1} > b_n)$. $f_A(a_n) = f_B(a_n) = \frac{1}{4}$, $f_A(b_n) = \frac{1}{2}$, $f_B(b_n) = \frac{1}{8}$, and all the elements of A will be contained in segments $(a_n, b_n]$, but elements of B will be in segments $(b_n, a_{n+1}]$. If we set $a_1 = 4$, $b_n = 2a_n$, $a_{n+1} = 2b_n$, we be able to build it easily. □

4 Further investigation

Theorem 6. *For any* $\lambda \in [0, 1]$ *there exists a set* $E \subset \mathbb{N}$ *such that* $\mu_1(E) = \lambda$.

Proof. The case $\lambda = 0$ is trivial, assume $\lambda > 0$. Then there exist a monotonically increasing sequence $\{\frac{p_n}{q_n}{}_{n\in\mathbb{N}}\}$ of rational numbers such that $\lim\limits_{n\to+\infty} \frac{p_n}{q_n} = \lambda$. Let k_n be such natural sequence that $k_1 = 1$ and $k_{n+1}q_{n+1} - k_nq_n > k_{n+1}p_{n+1} - k_np_n$. k_n always exist since $p_n < q_n$. Let $f_E(n) = \frac{\#(E\cap[1,n])}{n}$, let $a_0 = 1, a_n = k_nq_n, n \in \mathbb{N}$ and let $c_n = a_{n+1} - (k_{n+1}p_{n+1} - k_np_n)$, where $k_0, p_0 = 0$. Let a set E on a segment $(a_n, a_{n+1}]$ be defined in the following way. It includes the whole natural segment $(c_n, a_{n+1}]$ and excludes natural segment $(a_n, c_n]$. Then it is easy to see that on segment $[1, a_n]$ the set E contains k_np_n elements. Then $f_E(a_n) = \frac{p_n}{q_n}$, when $x \in (c_n, a_{n+1}) \cap \mathbb{N}, f_E(x) < f_E(a_n)$ and when $x \in (a_n, c_n] \cap \mathbb{N}, f_E(x) < f_E(a_n) < f_E(a_{n+1})$. Thus, $\sup\limits_{n\in(a_n,a_{n+1}]\cap\mathbb{N}} f_E(n) = \frac{p_{n+1}}{q_{n+1}}$. Then

$$\mu_1(E) = \limsup_{n\to+\infty} \frac{\#\left(E \cap [1, n]\right)}{n} = \lambda.$$

□

References

[1] *The theory of the Riemann Zeta-Function*, E. C. Titchmarsh, Oxford 1951