Russia #1, LCME Saint-Petersburg

Problem 1: Generilizing Perfectness

Abstract

We did point 1.a, first half of point 1.b and obtained some results in last part of point 1.b. We did point 2, 3, 4 and point 5a.

Remark.

1. As when $n = 1$ the sum $\sum\limits_{\substack{d|n \\ 1\le d\le n-1}} f(d)$ doesn't make sence, we don't consider this case.

1. If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_k^{\alpha_k}$, $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)...(\alpha_k + 1)$
2. $\forall n \ge 2$ $\sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d) \ge 2\tau(n) - 3$
3. If $\varphi(n)$ - Euler's totient function, $\sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \varphi(d) = n$

1 First problem

1.1 Question: Prove that a natural number $n \ge 1$ is $\tau - perfect$ $\Longleftrightarrow$ n is a square of a prime.

Proof. $\Longrightarrow$:

Let $n \in \mathbb{N}$ be a $\tau - perfect$. Then $\tau(n) = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d) \ge 2\tau(n) - 3 \Longrightarrow \tau(n) \le 3$. Then $n = p^2$ where p - prime.

$\Longleftarrow$:

Let $n = p^2$ where p is prime. Then $\tau(n) = 3 = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d)$. □

1.2 Question 1: Find all $f - perfect$ natural numbers $n \ge 1$ for the function $f(n) = \tau(n) - 1$.

Theorem 1. *If $n \in \mathbb{N}$, n - $f - perfect$ $\Longleftrightarrow$ $n = p^3$ where p - prime.*

Proof. $\Longrightarrow$:

Let $n \in \mathbb{N}$ be a $f - perfect$. Then $f(n) = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} f(d) \Longrightarrow \tau(n) - 1 = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d) - (\tau(n) - 1) \Longrightarrow 2\tau(n) - 2 = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d) \ge 2\tau(n) - 3$. Then $\exists! d_1$ - divisor of n that $d_1 = 1$; $\exists! d_2$ - divisor of n that $d_2 = p^2$,where p is prime and all the other divisors are prime. Let d_3, d_4 be divisors of n, where d_3, d_4 are primes and $d_3 \ne d_4$, then $d_3 d_4$ is divisor of n. Controversial. Then $n = p^3$.

$\Longleftarrow$:

Let $n = p^3$ where p is prime. Then $f(n) = 3 = f(1) + f(p) + f(p^2) = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} f(d)$. □

1.3 Question 2: Find as many values of $k \in \mathbb{Z}$ find all $f - perfect$ natural numbers $n \ge 1$ for the function $f(n) = \tau(n) + k$.

Theorem 2. *If $k \ge 1 \Longrightarrow \forall n \in \mathbb{N}$ n is **not** $f - perfect$.*

Proof. Let $k \ge 1$ and $\exists n \in \mathbb{N}$ such that n - $f - perfect$. Then $\tau(n) + k = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} (\tau(d) + k) \Longrightarrow \tau(n) + k = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d) + (\tau(n) - 1)k \Longrightarrow \tau(n)(1 - k) + 2k = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d) \ge 2\tau(n) - 3 \Longrightarrow \tau(n)(1 - k) + 2k \ge 2\tau(n) - 3 \Longrightarrow \frac{2k+3}{k+1} \ge \tau(n)$. Then $\tau(n) \le 2$, i.e. n is a prime $\Longrightarrow \tau(n) + k = \sum\limits_{\substack{d|n \\ 1\le d\le n-1}} \tau(d)$ that is k+2=k+1. ?!! □

Claim 3. Let $k = -2$, then if $n = p_1^3 p_2 \Longrightarrow n - f - perfect$ sand let $k = -5$, then if $n = p_1^4 p_2 p_3 \Longrightarrow$n - $f - perfect$.

2 Second problem

2.1 Question: Find all $f - perfect$ numbers n, where $f(n) = \varphi(n)$.

Theorem 4. *$n \in \mathbb{N}$ n - $\varphi - perfect \Longleftrightarrow n = 2^k$, where $k \in \mathbb{N}$*

Proof. $\Longrightarrow$:

Let n be a $\varphi - perfect \Longrightarrow \varphi(n) = \sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} \varphi(d) = n - \varphi(n) \Longrightarrow 2\varphi(n) = n$. Then $\forall d : 1 \le d \le n-1$ and $gcd(d, 2) = 1 \Rightarrow gcd(n, d) = 1$. Then $n = 2^k$, where $k \in \mathbb{N}$.

$\Longleftarrow$:

Let $n = 2^k$, where $k \in \mathbb{N}$. Then $\varphi(n) = \frac{n}{2} = 2^{k-1} = n - \varphi(n) = \sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} \varphi(d)$. □

3 Third problem

3.1 Question a: Prove that if k is a natural number such that $2^{k+1} - 2k - 1$ is a prime, then $n = 2^k(2^{k+1} - 2k - 1)$ is $f - perfect$ for $f(n) = n + 1$.

Proof. <see: 3.2 Question b> □

3.2 Question b: Find similar sufficient conditions for $f - perfectness$ for other polynimial functions of degree 1.

Theorem 5. *Let $f(n) = an + b$ and $k \in \mathbb{N}$, where $a|2kb$ and $t = 2^{k+1} + \frac{2kb}{a} - 1$ is prime. Then if $n = 2^k t$ $\Longrightarrow$ n - $f - perfect$.*

Proof. Let $n = 2^k t$. Then $\sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} f(d) = \sum\limits_{i=0}^{k} (a2^i + b) + \sum\limits_{i=0}^{k-1} (a2^i t + b) = (2k+1)b + (2^{k+1} - 1)a + (2^k - 1)at = 2kb + b + a2^{k+1} - a + a2^k t - at = a2^k t + b = an + b = f(n)$. □

4 Forth problem

4.1 Question: Let $f(n) = ln(n)$.Find all $f - perfect$ numbers n.

Theorem 6. *$n \in \mathbb{N}$ n - $ln - perfect \Longleftrightarrow n = p^3$ or $n = pq$, where p,q are primes.*

Proof. $\Longrightarrow$:

Let n be a $ln - perfect$. Then $ln(n) = \sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} ln(d) = ln(\prod\limits_{\substack{d|n \\ 1 \le d \le n-1}} d)$, that is $n = \prod\limits_{\substack{d|n \\ 1 \le d \le n-1}} d$ .Let $n = p^k m$, where p is prime and gcd(m,p)=1. Then denote by $s = \tau(m) \Longrightarrow \prod\limits_{\substack{d|n \\ 1 \le d \le n-1}} d = p^{\frac{sk(k+1)}{2} - k} M$, where gcd(M,p)=1. Then $p^k m = p^{\frac{sk(k+1)}{2} - k} M$, that is $4 = s(k+1)$.

1. s=1 and k=3. Then $n = p^3$,where p is prime. Let's check: $n = p^3 = 1 * p * p^2$.
2. s=2 and k=1. Then $n = pq$, where $p \ne q$ and p,q are primes. Let's check: $n = pq = 1 * p * q$.

$\Longleftarrow$: <see: Theorem 5$\Longrightarrow$: > □

5 Fifth problem

5.1 Question: Let $f(n) = (-1)^n$. Find all $f - perfect$ numbers n.

Theorem 7. *$n \in \mathbb{N}$ $n = p$ or $n = 4q$, where p, q are primes and gcd(p,2)=1$\Longleftrightarrow$ $n-$ $f - perfect$*

Proof. $\Longrightarrow$:

1. Let $n = p$, where p is prime and gcd(p,2)=1. Then $f(n) = -1$ and $\sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} f(d) = -1$.

2. Let $n = 4q$, where q is prime. Then $f(n) = 1$ and $\sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} f(d) = 1$.

$\Longleftarrow$:

Let n be a $f - perfect$.

1. $2|n$. $\forall d : d|n$, $f(d) = -1$. Then n has only one divisor, that is $n = p$, where p is prime and gcd(p,2)=1.

2. gcd(n,2)=1. Let $n = 2^k m$, where $gcd(m, 2) = 1$. Let's denote by s: $s = \tau(m)$. The number of even addends is equal to number of odd addends plus 1 in this sum $\sum\limits_{\substack{d|n \\ 1 \le d \le n-1}} f(d)$. That is $ks - 1 = s + 1 \Longrightarrow$ (k-1)s=2.

2.1 k=3, s=1. Then n=8. Let's check: <see Theorem 6$\Longrightarrow$:>

2.2 k=2, s=2. Then n=4p, where p is prime. <see Theorem 6$\Longrightarrow$:>. ∎