5. Stable Polygons

Abstract. We start this paper off by trying to classify "stable polygons" (subsets of the vertices of a regular n-gon having their center of gravity in the center of the polygon). In order to do so we regard the vertices of the polygon as roots of unity and identify the subsets with sum zero. First, we prove the linear independence of $p-1$ p-th roots of unity over $\mathbb{Q}$ thereby showing there is only one stable polygon (the whole polygon).

We then go on by determining "prime" linear combination of roots of unity which is a generalization of the concept of stable "polygon". It is shown that finding prime polygons is directly related to minimal linearly dependent sets. We then prove that the only prime polygons for $n = pq$ are the regular p- and$q$-gon concluding there are $2^p + 2^q - 2$ stable polygons for $n = pq$ (as p-and q-gons are mutually exclusive).

We close out the paper by giving further generalizations which are not yet proved.

Note: In this text we assume $0 \in \mathbb{N}$.

We regard the vertices of the n-gon as n-th roots of unity in the compley plane 0 being the center of the polygon.

Then a polygon is stable iff the sum of its vertices is zero.

Lemma 1. *For p prime the p-th cyclotomic polynomial is the minimal polynomial for the primitive p-th roots of unity.*

Proof. Every primitive p-th root of unity is a root of the p-th cyclotomic polynomial. Therefore the minimality follows from the irreducibility of the cyclotomic polynomial.

We investigate the polynomial $\frac{(x+1)^p-1}{(x+1)-1}$ which is irreducible iff the cyclotomic polynomial is irreducible (just shifting $x \to x+1$). We have

$$\begin{aligned}\frac{(x+1)^p-1}{(x+1)-1} &= \frac{(x^p + \binom{p}{1}x^{p-1} + \binom{p}{2}x^{p-2} + \cdots + \binom{p}{p-1}x + 1) - 1}{x} \\ &= x^{p-1} + \binom{p}{1}x^{p-2} + \binom{p}{2}x^{p-3} + \cdots + p\end{aligned}$$

Obviously $p \nmid 1$. Since p is prime $p | \binom{p}{i}$ for $1 \leq i \leq p-1$ and $p | p$ but $p^2 \nmid p$. Thus, the polynomial is irreducible due to Eisenstein's Criterion. □

Theorem 1. *For p prime $p-1$ of the $p-th$ roots of unity are linearly independent over $\mathbb{Q}$.*

Proof. Assume there are $p-1$ linearly dependent p-th roots of unity. Let ζ be a primitive p-th root of unity. W.l.o.g. (just rotate/multiply) we can assume 1 is not part of the aforementioned $p-1$ roots of unity. Since they are linearly dependent we have

$$\sum_{i=1}^{p-1} c_i \zeta^i = 0$$

$$\zeta \cdot \sum_{i=0}^{p-2} c_{i+1} \zeta^i = 0$$

for some c_i rational not all being zero and therefore ζ is a root of the polynom $\sum_{i=0}^{p-2} c_{i+1} x^i$ of degree at most $p-2$ which contradicts Lemma 1. □

Corollorary. *The one and only stable (and prime[see below]) polygon for p prime is the regular p-gon itself.*

Proving the linear independence over $\mathbb{Q}$ (which is more than originally required) suggests a generalization:

A *polygon* is now considered to be a linear combination of roots of unity over $\mathbb{Q}$ (and can indeed still be regarded as a polygon in the complex plane). A polygon is *stable* iff it is zero.

First, we note it is sufficient to consider linear combinations over $\mathbb{Z}$ as one can multiply a linear combination over $\mathbb{Q}$ with the least common denominator of the coefficients preserving the sum being zero.

Furthermore we note that for all natural $0 \leq k \leq n-1$ and a non-negative we have:

$$-a\zeta^k = a \sum_{i=0;i\neq k}^{n-1} \zeta^i$$

Using this relation we can convert a linear combination containing negative coefficients into a linear combination with only non-negative coefficients by substituting all terms with negative coefficient. Thus, it is sufficient to consider linear combinations over $\mathbb{N}$. It is worth noting that we still implicitly study the original problem.

We call a stable polygon $\sum_{i=0}^{n-1} a_i\zeta^i = 0$ *prime* iff there does not exist a stable polygon $\sum_{i=0}^{n-1} b_i\zeta^i = 0$ not all b_i being zero with $b_i \leq a_i$ for all i and $b_i < a_i$ for at least one i.

As all stable non-prime polygons can be split into a sum of prime polygons (otherwise being prime themselves) our remaining goal is to identify all prime polygons. The roots of unity with non-zero coefficients of a stable polygon are obviously linearly dependent. The polygon shall be called *linearly over-dependent* iff there is a linearly dependent proper subset of those linearly dependent roots of unity with non-zero coefficients.

Theorem 2. *A stable polygon is prime iff it is not linearly over-dependent and its non-zero coefficients are relatively prime.*

Proof. Assume there exists a stable polygon $a = \sum_{i=0}^{n-1} a_i\zeta^i = 0$ with $gcd(a_0, a_1, \cdots, a_{n-1}) = 1$ which is not linearly over-dependent and not prime. Then there exists a stable polygon $\sum_{i=0}^{n-1} b_i\zeta^i = 0$ not all b_i being zero with $b_i \leq a_i$ for all i and $b_i < a_i$ for at least one i.

W.l.o.g. let $b_0 \neq 0$. Now we consider

$$\begin{aligned} & b_0 \sum_{i=0}^{n-1} a_i\zeta^i - a_0 \sum_{i=0}^{n-1} b_i\zeta^i = 0 \\ \iff \quad & \sum_{i=1}^{n-1} (b_0 a_i - a_0 b_i)\zeta^i = 0 \end{aligned}$$

Since a is not linearly over-dependent we have $(b_0a_i - a_0b_i) = 0 \iff b_i = a_i \cdot \frac{b_0}{a_0}$ for all i. Since $gcd(a_0, a_1, \cdots, a_i, \cdots, a_{n-1}) = 1$ and $0 \neq b_0 \leq a_0$ we have $b_i = a_i$ which contradicts the existence of a $b_i < a_i$. $\square$

This Theorem gives us a powerful tool for the classification of prime polygons employed up next.

Theorem 3. *The only prime polygons for $n = pq$ with p, q distinct prime are the q regular p-gons and p regular q-gons.*

Proof. Assume there is a prime polygon a apart from the regular p,q-gon. Let ζ_p,ζ_q be a primitive p-th,q-th root of unity, respectively. As p and q are coprime we can use the Chinese Remainder Theorem and write a as

$$\begin{aligned} a &= \sum_{\substack{0\le i<p \\ 0\le j<q}} a_{ij}\zeta_p^i\zeta_q^j \\ &= \sum_{0\le i<p} \zeta_p^i \sum_{0\le j<q} a_{ij}\zeta_q^j \qquad = 0 \end{aligned}$$

By simple extension of Theorem 1 we have the linear independence of $p-1$ p-th roots of unity over $\mathbb{Q}(\zeta_q)$ and thus for every $1 \le j < q$ we have

$$\sum_{j=0}^{q-1} a_{0j}\zeta_q^j = \sum_{j=0}^{p-1} a_{ij}\zeta_q^j$$

$$\sum_{j=0}^{q-1} (a_{0j} - a_{ij})\zeta_q^j = 0$$

Because of Theorem 1 we have $(a_{0j} - a_{ij}) = d_j = const.$ for every i, j. If $d_j \neq 0$ for a j all the a_{0j} or all the a_{ij} are positive and thus contain a regular q-gon as a subset which contradicts a being prime. Therefore for a given i we have $a_{0j} = a_{ij}$. If all a_{0j} are zero then a is not a polygon. But if $a_{0j} \neq 0$ for a given j the a_{ij} contain the regular p-gon as a subset. Contradiction! □

Corollorary. *Just allowing the coefficients* 0 *and* 1 *we note that* p*- and* q*-gons are mutually exclusive thus allowing* $(2^p - 1) + (2^q - 1) = 2^p + 2^q - 2$ *distinct stable polygons (polygons are not empty).*

Unfortunately, it is not possible to generalize this Theorem for more than 2 primes, e.g. pqr (p,q,r prime) as shown by the following counter-example:

Counter-Example. *Let* ζ *be a primitive* 30*-th root of unity. Then*

$$\zeta^5+\zeta^6+\zeta^{12}+\zeta^{18}+\zeta^{24}+\zeta^{25} = (\zeta^0+\zeta^6+\zeta^{12}+\zeta^{18}+\zeta^{24})+(\zeta^5+\zeta^{15}+\zeta^{25})-(\zeta^0+\zeta^{15}) = 0+0-0 = 0$$

is a stable and prime polygon which is not a regular n*-gon.*

Outlook

The following conjecture has been derived by careful testing:

Conjecture. *The only stable prime polygons for* $n = p^a$ *are the regular* p*-gons.*

If this is true we can reuse the proof of Theorem 3 and we gain

Theorem. *The only stable prime polygons for* $n = p^a q^b$ *are the regular* p*- and* q*-gons.*

Additionally we found a way to generalize Theorem 3 and gain an answer to the whole problem:

Conjecture. *All stable polygons are a linear combination of regular* p_1*-,*p_2*-,*$\cdots$*-gons (the* p_i*s being the prime numbers).*