4. Isosceles Triangles

1. Let AA_1 be the internal bisector of $\angle CAB$ and BB_1 be the internal bisector of $\angle ABC$, such that $A_1 \in BC$ and $B_1 \in CA$. Suppose $\overline{AA_1} = \overline{BB_1}$. By Stewartï¿½s theorem we get
$\overline{AA_1} = \sqrt{\frac{\overline{A_1C}\cdot\overline{AB}^2+\overline{A_1B}\cdot\overline{AC}^2}{\overline{BC}} - \overline{A_1B}\cdot\overline{A_1C}}$
and $\overline{BB_1} = \sqrt{\frac{\overline{B_1C}\cdot\overline{AB}^2+\overline{B_1A}\cdot\overline{BC}^2}{\overline{AC}} - \overline{B_1A}\cdot\overline{B_1C}}$, thus
$\frac{\overline{A_1C}\cdot\overline{AB}^2+\overline{A_1B}\cdot\overline{AC}^2}{\overline{BC}} - \overline{A_1B}\cdot\overline{A_1C} = \frac{\overline{B_1C}\cdot\overline{AB}^2+\overline{B_1A}\cdot\overline{BC}^2}{\overline{AC}} - \overline{B_1A}\cdot\overline{B_1C}$. Due to the bisector theorem, we know that
$\overline{A_1B} = \frac{\overline{AB}\cdot\overline{BC}}{\overline{AB}+\overline{AC}}$, $\overline{A_1C} = \frac{\overline{AC}\cdot\overline{BC}}{\overline{AB}+\overline{AC}}$, $\overline{B_1C} = \frac{\overline{AC}\cdot\overline{BC}}{\overline{AB}+\overline{BC}}$, $\overline{B_1A} = \frac{\overline{AB}\cdot\overline{AC}}{\overline{AB}+\overline{BC}}$. By inserting these values into the equation above, we obtain
$\overline{AB}\cdot\overline{AC} - \overline{AB}\cdot\overline{AC}\cdot\overline{BC}\cdot\frac{\overline{BC}}{\left(\overline{AB}+\overline{AC}\right)^2} = \overline{AB}\cdot\overline{BC} - \overline{AB}\cdot\overline{AC}\cdot\overline{BC}\cdot\frac{\overline{AC}}{\left(\overline{AB}+\overline{BC}\right)^2}$
$\iff \frac{1}{\overline{BC}} - \frac{\overline{BC}}{\left(\overline{AB}+\overline{AC}\right)^2} = \frac{1}{\overline{AC}} - \frac{\overline{AC}}{\left(\overline{AB}+\overline{BC}\right)^2}$. Suppose $\overline{AC} > \overline{BC}$. Thus
$\frac{1}{\overline{BC}} > \frac{1}{\overline{AC}}$ and because $\overline{AB}+\overline{AC} > \overline{AB}+\overline{BC}$, $\frac{\overline{BC}}{\left(\overline{AB}+\overline{AC}\right)^2} < \frac{\overline{AC}}{\left(\overline{AB}+\overline{BC}\right)^2}$, hence
$\iff \frac{1}{\overline{BC}} - \frac{\overline{BC}}{\left(\overline{AB}+\overline{AC}\right)^2} > \frac{1}{\overline{AC}} - \frac{\overline{AC}}{\left(\overline{AB}+\overline{BC}\right)^2}$, which is a contradiction. $\overline{AC} < \overline{BC}$ leads to a contradiction, simultaneously. Therefore it is shown that $\overline{AC} = \overline{BC}$, so the triangle is isosceles.

3. If the triangle is isosceles the to external bisectors are equal, since the triangle is symmetric. So now assume that $\overline{A_1'A} = \overline{B_B'}$ Denote by α, β, γ the angles $\angle CAB, \angle ABC, \angle BCA$. Since AA_1' and BB_1' are external bisectors we conclude: $\angle BAA_1' = \frac{\pi}{2} - \frac{\alpha}{2}$ and $\angle B_1'BA = \frac{\pi}{2} - \frac{\alpha}{2}$ With the theorem of external angle we get: $\angle AB_1'B = \alpha + \frac{\beta}{2} - \frac{\pi}{2}$ and $\angle AA_1'B = \beta + \frac{\alpha}{2} - \frac{\pi}{2}$. Using the law of sine in $A_1'BA$ we get:

$$\frac{\sin\left(\beta + \frac{\alpha}{2} - \frac{\pi}{2}\right)}{\sin(\pi - \beta)} = \frac{\sin\left(\beta + \frac{\alpha}{2} - \frac{\pi}{2}\right)}{\sin(\beta)} = \frac{\overline{A_1'A}}{\overline{AB}} = \frac{\overline{B_1'B}}{\overline{AB}} = \frac{\sin\left(\alpha + \frac{\beta}{2} - \frac{\pi}{2}\right)}{\sin(\alpha)}$$

(The last equality holds because of the law of sine in $B_1'BA$). So we have:

$$\cos\left(\beta + \frac{\alpha}{2}\right)\sin\alpha = \cos\left(\alpha + \frac{\beta}{2}\right)\sin\beta$$

Using $2\cos x\cos y = \cos(x+y) + \cos(x-y)$ we get:

$$\frac{\sin\alpha}{\cos\frac{\alpha}{2}}\left(\cos(\alpha+\beta) + \cos\beta\right) = \frac{\sin\beta}{\cos\frac{\beta}{2}}\left(\cos(\alpha+\beta) + \cos\alpha\right)$$

Using $\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$ we conclude:

$$\left(\sin\frac{\alpha}{2}-\sin\frac{\beta}{2}\right)\cos(\alpha+\beta)=\sin\frac{\beta}{2}\cos\alpha-\sin\frac{\alpha}{2}\cos\beta$$

With $\cos(x)=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=1-2\sin^2\frac{x}{2}$

$$\sin\frac{\beta}{2}\cos\alpha-\sin\frac{\alpha}{2}\cos\beta=(\sin\frac{\beta}{2}-\sin\frac{\alpha}{2})(1-2\sin\frac{\alpha}{2}\sin\frac{\beta}{2})$$

Assume that $\alpha\neq\beta$ so we have $\sin\alpha\neq\sin\beta$ Hence:

$$\cos(\alpha+\beta)=2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}-1$$

Now Let x, y be $\frac{\alpha}{2},\frac{\beta}{2}$.

$$\cos(2x+2y)=\cos^2(x+y)-\sin^2(x+y)=2\cos^2(x+y)-1=2\sin x\sin y-1\Leftrightarrow\cos^2(x+y)-\frac{\cos(x+y)}{2}$$

This eqality has obviously solutions with $1\geq\cos(x-y)>\cos(x+y)>0$ So we can construct solutions with $0<x+y<\frac{\pi}{2}, x, y>0$. So of course α and β can be different, because we just used eqivivalent transformations. (Allways looking at the range of the values. **5.** From the theorem of sine we get $\frac{|CA_1|}{|A_1B|}=\frac{\frac{\sin(\frac{\alpha}{2})|AA_1|}{\sin(\gamma)}}{\frac{\sin(\frac{\alpha}{2})|AA_1|}{\sin(\beta)}}=\frac{\sin(\beta)}{\sin(\gamma)}=\frac{b}{c}$.

For the external bisector it is $\frac{\left|CA_1'\right|}{\left|A_1'B\right|}=\frac{\frac{\sin(90-\frac{\alpha}{2})|AA_1'|}{\sin(\gamma)}}{\frac{\sin(90+\frac{\alpha}{2})\left|AA_1'\right|}{\sin(\beta)}}=\frac{\sin(\beta)}{\sin(\gamma)}=\frac{b}{c}$.

Looking at the symmedians we get that $\sin(\angle A_2AC)=\sin(\angle BAA_0)=\frac{\sin(\angle AA_0B)|BC|}{2|AB|}$ and $\sin(\angle BAA_2)=\sin(\angle A_0AC)=\frac{\sin(\angle AA_0B)|BC|}{2|AC|}$.

Hence:

$\frac{|CA_2|}{|A_2B|}=\frac{\frac{\sin(\angle A_2AC)|AA_0|}{\sin(\gamma)}}{\frac{\sin(\angle BAA_2)|AA_0|}{\sin(\beta)}}=\frac{\sin(\beta)|AC|}{\sin(\gamma)|BC|}=\frac{b^2}{c^2}$

Now last but not least the exsymmedians:

It is $\angle CAA_2'=\gamma-2\angle A_!'AA_0'=\gamma-2(\gamma-(90-\frac{\alpha}{2}))=\beta$

Hence:

$\frac{\left|CA_2'\right|}{\left|A_2'B\right|}=\frac{\frac{\sin(\beta)|AA_2'|}{\sin(\gamma)}}{\frac{\sin(\alpha+\beta)\left|AA_2'\right|}{\sin(\beta)}}=\frac{\sin(\beta)^2}{\sin(\gamma)\sin(\alpha+\beta)}=\frac{\sin(\beta)^2}{\sin(\gamma)^2}=\frac{b^2}{c^2}$