10. Densities of Natural Subsets

Abstract

Our paper starts with answering to the questions of problems 1, 2, 3 and 4, mostly by estimating the terms. In problem 5, we evaluate the densities of the Fibonacci sequence. In problems 6 and 7, we construct some sets, which contradict the equality from the problem. In problem 8, we introduce a new density μ_4 and solve the problems 1, 2, 3 and 4 for it.

The employed methods are basic algebraic and analytic manipulations.

Problem 1

The term whose upper limit is searched at μ_1 and μ_2 is bounded. As is well-known, at least one convergent subsequence exists, so that the upper limit also exists. Proof of the boundedness:

$$\mu_1: \quad 0 \leq \frac{|E \cap [1;n]\,|}{n} \leq \frac{n}{n} = 1 \quad \forall n \in \mathbb{N}$$
$$\mu_2: \quad 0 \leq (1-x)\sum_{n\in E} x^n \leq (1-x)\sum_{n\in\mathbb{N}_0} x^n = (1-x)\frac{1}{1-x} = 1 \quad \forall x \in]0;1[$$

Lemma:

$$\limsup_{x\to 1^+}\,(x-1)\sum_{n\in\mathbb{N}}\frac{1}{n^x} = 1$$

Proof: Let be $f_x : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_x(y) = \frac{1}{y^x}$ and $x \in]1;\infty[$. Obviously, it is $f_x(y_0+1) \leq \int_{y_0}^{y_0+1} f_x(y)\mathrm{d}y \leq f_x(y_0) \quad \forall y_0 \in \mathbb{N}$ because of the monotony. Adding these two inequalities brings:

$$\int_1^\infty f_x(y)\mathrm{d}y \leq \sum_{n=1}^\infty \frac{1}{n^x},$$
$$\sum_{n=2}^\infty \frac{1}{n^x} \leq \int_1^\infty f_x(y)\mathrm{d}y \quad | +1$$
$$\Rightarrow \int_1^\infty f_x(y)\mathrm{d}y \leq \sum_{n=1}^\infty \frac{1}{n^x} \leq \int_1^\infty f_x(y)\mathrm{d}y + 1$$

Now, because of $\int_1^\infty \frac{1}{y^x}\mathrm{d}y = \frac{1}{x-1}$, it is:

$$\frac{1}{x-1} \leq \sum_{n\in\mathbb{N}}\frac{1}{n^x} \leq \frac{x}{x-1} \quad | \cdot (x-1)$$
$$\Leftrightarrow 1 \leq (x-1)\sum_{n\in\mathbb{N}}\frac{1}{n^x} \leq x \quad \forall x \in]1;\infty[$$
$$\Rightarrow \limsup_{x\to 1^+}\,(x-1)\sum_{n\in\mathbb{N}}\frac{1}{n^x} = 1$$

For μ_3, we get:

$$\begin{aligned}
0 \leq (x-1)\sum_{n\in E}\frac{1}{n^x} \leq (x-1)\sum_{n\in\mathbb{N}}\frac{1}{n^x} \quad \forall x\in]1;\infty[\\
\limsup_{x\to 1^+}(x-1)\sum_{n\in\mathbb{N}}\frac{1}{n^x} = 1 \\
\Rightarrow 0 \leq \limsup_{x\to 1^+}(x-1)\sum_{n\in E}\frac{1}{n^x} \leq 1
\end{aligned}$$

So, $\mu_3(E)$ is also well-defined.

Problem 2

For a finite set E, it is:

$$\begin{aligned}
0 \leq \limsup_{n\to\infty}\frac{|E\cap[1;n]|}{n} \leq \limsup_{n\to\infty}\frac{|E|}{n} = 0 \\
\limsup_{x\to 1^-}(1-x)\sum_{n\in E}x^n = (1-1)\sum_{n\in E}1^n = 0 \\
\limsup_{x\to 1^+}(x-1)\sum_{n\in E}\frac{1}{n^x} = (1-1)\sum_{n\in E}\frac{1}{n^1} = 0 \\
\Rightarrow \mu_1(E) = \mu_2(E) = \mu_3(E) = 0
\end{aligned}$$

Problem 3

$$\begin{aligned}
\mu_1(E) &= \limsup_{n\to\infty}\frac{|\{k\in\mathbb{N}_0\,|\,a+kd\leq n\}|}{n} = \limsup_{n\to\infty}\frac{|\{k\in\mathbb{Z}\,|\,0\leq k\leq\frac{n-a}{d}\}|}{n} \\
&= \limsup_{n\to\infty}\frac{\left\lfloor\frac{n-a}{d}\right\rfloor+1}{n}
\end{aligned}$$

Furthermore, it is:

$$\frac{1}{d}+\frac{-\frac{a}{d}+1}{n} = \frac{\frac{n-a}{d}+1}{n} \leq \frac{\left\lfloor\frac{n-a}{d}\right\rfloor+1}{n} < \frac{\frac{n-a}{d}+2}{n} = \frac{1}{d}+\frac{-\frac{a}{d}+2}{n}.$$

Both boundaries converge to $\frac{1}{d}$, so that we get:

$$\mu_1(E) = \limsup_{n\to\infty}\frac{\left\lfloor\frac{n-a}{d}\right\rfloor+1}{n} = \frac{1}{d}.$$

$$\mu_2(E) = \limsup_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} x^{a+kd} = \limsup_{x\to 1^-} (1-x)x^a \sum_{k=0}^{\infty} \left(x^d\right)^k = \limsup_{x\to 1^-} (1-x)\frac{x^a}{1-x^d}$$
$$= \limsup_{x\to 1^-} \frac{(1-x)x^a}{(1-x)(1+x+x^2+\ldots+x^{d-1})} = \frac{1^a}{1+1+1^2+\ldots+1^{d-1}} = \frac{1}{d}$$

Obviously it is, such as in the proof of problem 1, because of the monotony of f_x:

$$f_x(y_0+\frac{a}{d}+1) \le \int_{y_0+\frac{a}{d}}^{y_0+\frac{a}{d}+1} f_x(y)\mathrm{d}y \le f_x(y_0+\frac{a}{d}) \quad \forall y_0 \in \mathbb{N}_0.$$

By adding these two inequalities, we get:

$$\int_{\frac{a}{d}}^{\infty} f_x(y)\mathrm{d}y \le \sum_{n=0}^{\infty} \frac{1}{(n+\frac{a}{d})^x},$$
$$\sum_{n=1}^{\infty} \frac{1}{(n+\frac{a}{d})^x} \le \int_{\frac{a}{d}}^{\infty} f_x(y)\mathrm{d}y \quad \mid + \frac{1}{(\frac{a}{d})^x}$$
$$\Rightarrow \int_{\frac{a}{d}}^{\infty} f_x(y)\mathrm{d}y \le \sum_{n=0}^{\infty} \frac{1}{(n+\frac{a}{d})^x} \le \int_{\frac{a}{d}}^{\infty} f_x(y)\mathrm{d}y + \frac{d^x}{a^x}$$

With $\int_{\frac{a}{d}}^{\infty} \frac{1}{y^x}\mathrm{d}y = \frac{d^{x-1}}{(x-1)a^{x-1}}$, we get:

$$\frac{d^{x-1}}{(x-1)a^{x-1}} \le \sum_{n\in\mathbb{N}_0} \frac{1}{(n+\frac{a}{d})^x} \le \frac{d^{x-1}}{(x-1)a^{x-1}} + \frac{d^x}{a^x}$$
$$= \frac{d^{x-1}a + d^x(x-1)}{(x-1)a^x} = \frac{d^{x-1}\left(1+\frac{d}{a}(x-1)\right)}{(x-1)a^{x-1}} \quad \mid \cdot \frac{x-1}{d^{x-1}}$$
$$\Leftrightarrow \frac{1}{a^{x-1}} \le \frac{x-1}{d^{x-1}} \sum_{n\in\mathbb{N}_0} \frac{1}{(n+\frac{a}{d})^x} \le \frac{1+\frac{d}{a}(x-1)}{a^{x-1}}$$

For $x \to 1^+$, both boundaries converge to 1, so we also get

$$\limsup_{x\to 1^+} \frac{x-1}{d^{x-1}} \sum_{n\in\mathbb{N}_0} \frac{1}{\left(n+\frac{a}{d}\right)^x} = 1.$$

Furthermore, we get:

$$\begin{aligned}\mu_3(E) &= \limsup_{x\to 1^+}(x-1)\sum_{n\in\mathbb{N}_0}\frac{1}{(nd+a)^x} = \limsup_{x\to 1^+}\frac{x-1}{d^x}\sum_{n\in\mathbb{N}_0}\frac{1}{\left(n+\frac{a}{d}\right)^x}\\ &= \frac{1}{d}\cdot\limsup_{x\to 1^+}\frac{x-1}{d^{x-1}}\sum_{n\in\mathbb{N}_0}\frac{1}{\left(n+\frac{a}{d}\right)^x} = \frac{1}{d}.\end{aligned}$$

Problem 4

Let the x that is used for the construction of the set be called y. Let x stand for the variable in the lim sup at μ_2 and μ_3 in the following. For $y = 1$, $E = \{1\}$ is a finite set and we get $\mu_1(E) = \mu_2(E) = \mu_3(E) = 0$ because of the solution of problem 2. Let us look at the case $y > 0$ in the following.

$$\begin{aligned}\mu_1(E) \le \limsup_{n\to\infty}\frac{|\{k\in\mathbb{N}_0\,|\,y^k\le n\}|}{n} \le \limsup_{n\to\infty}\frac{\log_y n+1}{n} = 0\\ \Rightarrow \mu_1(E) = 0\end{aligned}$$

Let be for all positive integers d: $E_d := \{nd\,|\,n\in\mathbb{N}\}$. As is well-known, it is $\lim\limits_{n\to\infty}\frac{nd}{y^n} = 0$, so $\frac{y^n}{nd}$ and also $\frac{[y^n]}{nd}$ are arbitrary great for sufficient great n, so particularly, it is $[y^n] > nd$ for sufficient great n. So let $m(d)$ a positive integer, such that it is $[y^k] > kd$ for all $k > m(d)$. Then we get:

$$\begin{aligned}\limsup_{x\to 1^-}(1-x)\sum_{n=0}^{m(d)}x^{[y^n]} = 0 = \limsup_{x\to 1^-}(1-x)\sum_{n=1}^{m(d)}x^{nd},\\ \limsup_{x\to 1^-}(1-x)\sum_{n=m(d)+1}^{\infty}x^{[y^n]} \le \limsup_{x\to 1^-}(1-x)\sum_{n=m(d)+1}^{\infty}x^{nd},\\ \limsup_{x\to 1^+}(x-1)\sum_{n=0}^{m(d)}\frac{1}{[y^n]^x} = 0 = \limsup_{x\to 1^+}(x-1)\sum_{n=1}^{m(d)}\frac{1}{(nd)^x},\\ \limsup_{x\to 1^+}(x-1)\sum_{n=m(d)+1}^{\infty}\frac{1}{[y^n]^x} \le \limsup_{x\to 1^+}(x-1)\sum_{n=m(d)+1}^{\infty}\frac{1}{(nd)^x}\end{aligned}$$

Adding the first two and the last two (in-)equalities brings (in due consideration of the solution of problem 3):

$$\begin{aligned}0 \le \mu_2(E) &= \limsup_{x\to 1^-} (1-x)\sum_{k\in E} x^k \le \limsup_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} x^{[y^n]}\\ &\le \limsup_{x\to 1^-}(1-x)\sum_{n=1}^{\infty} x^{nd} = \mu_2(E_d) = \frac{1}{d}\\ 0 \le \mu_3(E) &= \limsup_{x\to 1^+}(x-1)\sum_{k\in E}\frac{1}{k^x} \le \limsup_{x\to 1^+}(x-1)\sum_{n=0}^{\infty}\frac{1}{[y^n]^x}\\ &\le \limsup_{x\to 1^+}(x-1)\sum_{n=1}^{\infty}\frac{1}{(nd)^x} = \mu_3(E_d) = \frac{1}{d}\end{aligned}$$

These two inequalities are true for all positive integers d and the upper boundary converges to 0 if $d \to \infty$. So we get $\mu_2(E) = \mu_3(E) = 0$.

Problem 5

Let the sequence (s_n) be the Fibonacci sequence with $s_0 = s_1 = 1$, $s_2 = 2$, ... and let E be the set of all members from s_1 (in the following, we do not look at s_0). As is well-known, it is $\lim\limits_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{1+\sqrt{5}}{2} > 1{,}6$, so finitely many quotients of two sequent Fibonacci numbers are less than 1,6, so that their minimum exists, which is obviously greater than 1. Let a be a real number being less than this minimum and less than $\sqrt{2}$, but greater than 1. So we have $a < \frac{s_{n+1}}{s_n}\ \forall n \in \mathbb{N}$, $[a] = s_1$, $[a^2] \le a^2 \le s_2$ and, with induction, we get $[a^n] \le a^n \le s_n$. Then it is:

$$\begin{aligned}0 \le &\mu_1(E) = \limsup_{n\to\infty}\frac{\{k\in\mathbb{N}\,|\,s_k \le n\}}{n} \le \limsup_{n\to\infty}\frac{|\{k\in\mathbb{N}\,|\,a^k\le n\}|}{n} \le \limsup_{n\to\infty}\frac{\log_a n + 1}{n} = 0\\ \Rightarrow &\mu_1(E) = 0\end{aligned}$$

Furthermore, let A be the set of all $[a^n]$, including $[a^0]$. Let K be the set of $n \in \mathbb{N}_0$ for that a $k \in \mathbb{N}$ with $k < n$ exists, such that $[a^k] = [a^n]$. This set is finite because, for sufficient great n, the difference $a^{n+1} - a^n$ is greater than 1. Then we get (in due consideration of the solution of problem 4):

$$\begin{aligned}
0 \le \mu_2(E) &\le \limsup_{x\to 1^-} (1-x) \sum_{n\in\mathbb{N}_0} x^{[a^n]} \\
&= \limsup_{x\to 1^-} (1-x) \sum_{n\in K} x^{[a^n]} + \limsup_{x\to 1^-} (1-x) \sum_{n\in\mathbb{N}_0\setminus K} x^{[a^n]} = 0 + \mu_2(A) = 0, \\
0 \le \mu_3(E) &\le \limsup_{x\to 1^+} (x-1) \sum_{n\in\mathbb{N}_0} \frac{1}{[a^n]^x} \\
&= \limsup_{x\to 1^+} (x-1) \sum_{n\in K} \frac{1}{[a^n]^x} + \limsup_{x\to 1^+} (x-1) \sum_{n\in\mathbb{N}_0\setminus K} \frac{1}{[a^n]^x} = 0 + \mu_3(A) = 0 \\
\Rightarrow \mu_2(E) &= \mu_3(E) = 0
\end{aligned}$$

Subsets with interesting densities will be constructed in the solutions of the problems 6 and 7.

Problem 6

Let be $E_1 := \mathbb{N} \cap \bigcup_{k=0}^{\infty} [2^{2k}; 2^{2k+1}-1]$. Now we look at the sequence $(b_n) = \left(\frac{|E_1\cap[1;n]|}{n}\right)$. If and only if $n \in E_1$, it is $b_{n-1} \le b_n$, otherwise, it is $b_{n-1} > b_n$. As we look at the upper limit, it is enough to look at the subsequence of (b_n), such that, for all choosen indexes n, it is $n \in E_1, n+1 \notin E_1$. Obviously, these are exactly the numbers with the represantation $n = 2^{2k+1}-1$ for all $k \in \mathbb{N}_0$. For these n, we calculate b_n and let n go to ∞. We partition the set $E_1 \cap [1; 2^{2k+1}-1]$ into $F_l := \mathbb{N} \cap [2^{2l}; 2^{2l+1}-1]$ for $l \in \mathbb{N} \cap [0; k]$. Then it is $|F_l| = 2^{2l+1} - 2^{2l} = 2^{2l}$. So we get

$$|E_1 \cap [1; 2^{2k+1}-1]| = \sum_{l=0}^{k} |F_l| = \sum_{l=0}^{k} 2^{2l} = \frac{2^{2k+2}-1}{3}.$$

Furthermore, it is

$$b_{2^{2k+1}-1} = \frac{\frac{2^{2k+2}-1}{3}}{2^{2k+1}-1} = \frac{2^{2k+2}-2}{3\cdot(2^{2k+1}-1)} + \frac{1}{3\cdot(2^{2k+1}-1)} = \frac{2}{3} - \frac{1}{3\cdot(2^{2k+1}-1)}$$
$$\Rightarrow \lim_{k\to\infty} b_{2^{2k+1}-1} = \frac{2}{3}.$$

So the result is $\mu_1(E_1) = \frac{2}{3}$.

Now we do the same for the set $E_2 := \mathbb{N}\setminus E_1$. We look at the sequence $(c_n) = \left(\frac{|E_2\cap[1;n]|}{n}\right)$. Here, for exactly those numbers with the representation $n = 2^{2k}-1$ with $k \in \mathbb{N}$, it is

$n \in E_2, n+1 \notin E_2$. Now we partition $E_2 \cap [1; 2^{2k}-1]$ into $G_l := \mathbb{N} \cap [2^{2l-1}; 2^{2l}-1]$ for $l \in \mathbb{N} \cap [1; k]$. Then we get $|G_l| = 2^{2l} - 2^{2l-1} = 2^{2l-1}$ and

$$|E_2 \cap [1; 2^{2k}-1]| = \sum_{l=1}^{k} |G_l| = 2 \cdot \sum_{l=0}^{k-1} 2^{2l} = \frac{2 \cdot (2^{2k}-1)}{3}.$$

Furthermore, it is

$$c_{2^{2k}-1} = \frac{\frac{2\cdot(2^{2k}-1)}{3}}{2^{2k}-1} = \frac{2}{3} \quad \Rightarrow \lim_{k\to\infty} c_{2^{2k}-1} = \frac{2}{3}$$

and, this way, we also get $\mu_1(E_2) = \frac{2}{3}$. So, the equality $\mu_1(A \cup B) + \mu_1(A \cap B) = \mu_1(A) + \mu_1(B)$ is not for all $A, B \subseteq \mathbb{N}$ true because

$$\mu_1(E_1 \cup E_2) + \mu_1(E_1 \cap E_2) = \mu_1(\mathbb{N}) + \mu_1(\{\}) = 1 \neq \frac{4}{3} = \mu_1(E_1) + \mu_1(E_2).$$

Problem 7

Lemma: For each $a \in \mathbb{N}$, each $y \in]0;1[$ and each $\varepsilon \in]0;1[$, there is a $x(a,y,\varepsilon) \in]y;1[$ (called x_1 in the following) and also a $b(a, x_1, \varepsilon) \in \mathbb{N}$ (called b_1 in the following) with $b_1 > a$, so that

$$(1-x_1)\sum_{n=a}^{b_1} x_1^n > 1-\varepsilon,$$

$$(1-x_1)\sum_{n=a}^{b_1} \frac{1}{n^{2-x_1}} > 1-\varepsilon.$$

Proof: The following two terms are bounded, so it is

$$0 < \sum_{n=0}^{a-1} x^n \leq \sum_{n=0}^{a-1} 1^n = a, \quad 0 < \sum_{n=1}^{a-1} \frac{1}{n^{2-x}} \leq \sum_{n=1}^{a-1} 1 = a-1 \quad \forall x \in]y;1[.$$

This is why we can choose a $x_1 \in]y;1[$ that great, so that it is

$$0 < (1-x_1)\sum_{n=0}^{a-1} x_1^n < \frac{\varepsilon}{2},\ 0 < (1-x_1)\sum_{n=1}^{a-1} \frac{1}{n^{2-x_1}} < \frac{\varepsilon}{2}.$$

Now, for all $b \in \mathbb{N}$ with $b > a$, we have

$$\sum_{n=b+1}^{\infty} x_1^n = \sum_{n=0}^{\infty} x_1^n - \sum_{n=0}^{b} x_1^n = \frac{1}{1-x_1} - \frac{1-x_1^{b+1}}{1-x_1} = \frac{x_1^{b+1}}{1-x_1}.$$

This is why we can choose a b_0 that great that

$$(1-x_1)\sum_{n=b+1}^{\infty} x_1^n < \frac{\varepsilon}{2} \forall b \geq b_0.$$

Furthermore, we can choose a sufficient great $b_1 \geq b_0$, so that

$$(1-x_1)\sum_{n=b_1+1}^{\infty} \frac{1}{n^{2-x_1}} < \frac{\varepsilon}{2}.$$

With these two inequalities and $(1-x_1)\sum_{n=1}^{\infty} \frac{1}{n^{2-x_1}} \geq 1$ (look at the proof of problem 1) we get what we need:

$$\begin{aligned}(1-x_1)\sum_{n=a}^{b_1} x_1^n =&(1-x_1)\sum_{n=0}^{\infty} x_1^n - (1-x_1)\sum_{n=0}^{a-1} x_1^n - (1-x_1)\sum_{n=b_1+1}^{\infty} x_1^n > \\ &1-\frac{\varepsilon}{2}-\frac{\varepsilon}{2} = 1-\varepsilon, \\ (1-x_1)\sum_{n=a}^{b_1} \frac{1}{n^{2-x_1}} =&(1-x_1)\sum_{n=1}^{\infty} \frac{1}{n^{2-x_1}} - (1-x_1)\sum_{n=1}^{a-1} \frac{1}{n^{2-x_1}} - (1-x_1)\sum_{n=b+1}^{\infty} \frac{1}{n^{2-x_1}} > \\ &1-\frac{\varepsilon}{2}-\frac{\varepsilon}{2} = 1-\varepsilon.\end{aligned}$$

Now let be $\varepsilon \in]0;1[$ and let be $(y_n)_{n\in\mathbb{N}}$ a sequence that converges to 1 and for whose members it is $0 < y_n < 1$. Now we construct the sequences (a_n) and (x_n), where $a_1 := 1$ and then, alternately, $x_n := x(a_n, y_n, \varepsilon)$ and $a_{n+1} := b(a_n, x_n, \varepsilon) + 1$. Then by definition (x_n) is also a sequence that converges to 1 and whose members are in $]0;1[$. Now let be $E_\varepsilon := \mathbb{N} \cap \bigcup_{k=0}^{\infty} [a_{2k}, a_{2k+1} - 1]$ and $E'_\varepsilon := \mathbb{N} \backslash E_\varepsilon$, while $E'_\varepsilon = \mathbb{N} \cap \bigcup_{k=0}^{\infty} [a_{2k+1}, a_{2k+2} - 1]$. Now we want to prove that $\min(\mu_2(E_\varepsilon), \mu_2(E'_\varepsilon), \mu_3(E_\varepsilon), \mu_3(E'_\varepsilon)) > 1 - \varepsilon$. It is enough to prove that there is a sequence of x that converges to 1 (respectively from above or from below), such that the term is greater than $1-\varepsilon$. Let us start with $\mu_2(E_\varepsilon)$. We take the subsequence $(x_{2k})_{k\in\mathbb{N}}$. For each x_{2k}, the set $\mathbb{N} \cap [a_{2k}; b(a_{2k}, x_{2k}, \varepsilon)]$ is a subset of E_ε, so that we get $(1-x_{2k}) \sum_{n\in E_\varepsilon} x_{2k}^n > (1-x_{2k}) \sum_{n=a_{2k}}^{b(a_{2k}, x_{2k}, \varepsilon)} x_{2k}^n > 1-\varepsilon$. The other three cases are analogously to prove:

$$\mu_2(E'_\varepsilon), (x_{2k+1})_{k\in\mathbb{N}} : \mathbb{N}\cap[a_{2k+1}; b(a_{2k+1}, x_{2k+1}, \varepsilon)] \subset E'_\varepsilon$$
$$\Rightarrow (1-x_{2k+1})\sum_{n\in E'_\varepsilon} x_{2k+1}^n > (1-x_{2k+1})\sum_{n=a_{2k+1}}^{b(a_{2k+1},x_{2k+1},\varepsilon)} x_{2k+1}^n > 1-\varepsilon,$$
$$\mu_3(E_\varepsilon), (2-x_{2k})_{k\in\mathbb{N}} : \mathbb{N}\cap[a_{2k}; b(a_{2k}, x_{2k}, \varepsilon)] \subset E_\varepsilon$$
$$\Rightarrow ((2-x_{2k})-1)\sum_{n\in E_\varepsilon}\frac{1}{n^{2-x_{2k}}} > (1-x_{2k})\sum_{n=a_{2k}}^{b(a_{2k},x_{2k},\varepsilon)}\frac{1}{n^{2-x_{2k}}} > 1-\varepsilon,$$
$$\mu_3(E'_\varepsilon), (2-x_{2k+1})_{k\in\mathbb{N}} : \mathbb{N}\cap[a_{2k+1}; b(a_{2k+1}, x_{2k+1}, \varepsilon)] \subset E'_\varepsilon$$
$$\Rightarrow ((2-x_{2k+1})-1)\sum_{n\in E'_\varepsilon}\frac{1}{n^{2-x_{2k+1}}} > (1-x_{2k+1})\sum_{n=a_{2k+1}}^{b(a_{2k+1},x_{2k+1},\varepsilon)}\frac{1}{n^{2-x_{2k+1}}} > 1-\varepsilon$$

If we assume that the equality for μ_2 and μ_3 is true, inserting brings

$$1 = \mu_2(\mathbb{N}) + \mu_2(\{\}) = \mu_2(E_\varepsilon \cup E'_\varepsilon) + \mu_2(E_\varepsilon \cap E'_\varepsilon) = \mu_2(E_\varepsilon) + \mu_2(E'_\varepsilon) > 2 - 2\varepsilon,$$
$$1 = \mu_3(\mathbb{N}) + \mu_3(\{\}) = \mu_3(E_\varepsilon \cup E'_\varepsilon) + \mu_3(E_\varepsilon \cap E'_\varepsilon) = \mu_3(E_\varepsilon) + \mu_3(E'_\varepsilon) > 2 - 2\varepsilon$$

These two inequalities implicate that $\frac{1}{2} < \varepsilon$, though we also can choose $\varepsilon \le \frac{1}{2}$, and then none of the two equalities is true.

Problem 8

We introduce the density μ_4 with

$$\mu_4(E) := \limsup_{x\to 1^+}(x-1)\sum_{n=1}^{\infty}\frac{|E\cap[1;n]|}{n^{x+1}}.$$

It is well-defined because all terms are positive and with $|E\cap[1;n]| \le n$ we get

$$0 \le \limsup_{x\to 1^+}(x-1)\sum_{n=1}^{\infty}\frac{|E\cap[1;n]|}{n^{x+1}} \le \limsup_{x\to 1^+}(x-1)\sum_{n=1}^{\infty}\frac{n}{n^{x+1}}$$
$$= \limsup_{x\to 1^+}(x-1)\sum_{n=1}^{\infty}\frac{1}{n^x} = 1 \quad \text{(due to lemma from problem 1)}$$

Now we want to calculate the density of a finite set, of an arithmetic progression and of the integral parts of a geometric one.

Let E be a finite set. Then we get

$$0 \leq \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{|E\cap [1;n]|}{n^{x+1}} \leq \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{|E|}{n^{x+1}} = |E| \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{1}{n^{x+1}}.$$

The sum converges to a finite value and $(x-1)$ to 0, so $0 \leq \mu_4(E) \leq 0 \Rightarrow \mu_4(E) = 0$.

Now let be $E = \{a+dn | n \in \mathbb{N}_0\}$, with $a, d \in \mathbb{N}$. Now we get (similar to the solution of problem 3):

$$\mu_4(E) = \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{|E\cap [1;n]|}{n^{x+1}} = \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{\left\lfloor \frac{n-a}{d} \right\rfloor + 1}{n^{x+1}}.$$

Furthermore, we have

$$\frac{1}{dn^x} + \frac{-\frac{a}{d}+1}{n^{x+1}} = \frac{\frac{n-a}{d}+1}{n^{x+1}} \leq \frac{\left\lfloor \frac{n-a}{d} \right\rfloor + 1}{n^{x+1}} < \frac{\frac{n-a}{d}+2}{n^{x+1}} = \frac{1}{dn^x} + \frac{-\frac{a}{d}+2}{n^{x+1}}.$$

This way, we get

$$\begin{aligned} &\frac{1}{d} \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{1}{n^x} + \left(-\frac{a}{d}+1\right) \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{1}{n^{x+1}} \\ \leq \mu_4(E) \leq &\frac{1}{d} \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{1}{n^x} + \left(-\frac{a}{d}+2\right) \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{1}{n^{x+1}}. \end{aligned}$$

In both boundaries, the sum in the second summand converges to a finite value, $(x-1)$ converges to 0, so that the second summand falls away. The first summand converges due to the lemma in problem 1 to $\frac{1}{d}$, so that we get $\mu_4(E) = \frac{1}{d}$.

Finally, let be $E = \{\lfloor y^n \rfloor | n \in \mathbb{N}_0\}$, where $y > 1$ (otherwise, $|E|$ is finite and $\mu_4(E) = 0$). Then we get:

$$\mu_4(E) = \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{|E\cap [1;n]|}{n^{x+1}} \leq \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{\log_y n + 1}{n^{x+1}}.$$

Now we want to prove that, for sufficient great n, it is:

$$\begin{aligned} \log_y n < \sqrt{n} \quad &| \, y^{()} \\ \Leftrightarrow n < y^{\sqrt{n}} \quad &| \, m \to \sqrt{n} \\ \Leftrightarrow m^2 < y^m& \end{aligned}$$

This is a well-known fact now. Now let be n_0 a positive integer, such that for all $n > n_0$ it is $\log_y n < \sqrt{n}$. Then we get

$$\begin{aligned}\mu_4(E) &\leq \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{\infty} \frac{\log_y n + 1}{n^{x+1}} \\ &= \limsup_{x\to 1^+} (x-1) \sum_{n=1}^{n_0} \frac{\log_y n + 1}{n^{x+1}} + \limsup_{x\to 1^+} (x-1) \sum_{n=n_0+1}^{\infty} \frac{\log_y n + 1}{n^{x+1}}.\end{aligned}$$

The sum in the second summand converges to a finite value, so it is zero. This way, we get

$$\begin{aligned}\mu_4(E) &\leq \limsup_{x\to 1^+} (x-1) \sum_{n=n_0+1}^{\infty} \frac{\log_y n}{n^{x+1}} + \limsup_{x\to 1^+} (x-1) \sum_{n=n_0+1}^{\infty} \frac{1}{n^{x+1}} \\ &\leq \limsup_{x\to 1^+} (x-1) \sum_{n=n_0+1}^{\infty} \frac{\sqrt{n}}{n^{x+1}} + 0 \\ &= \limsup_{x\to 1^+} (x-1) \sum_{n=n_0+1}^{\infty} \frac{1}{n^{x+\frac{1}{2}}} = 0 \\ &\Rightarrow \mu_4(E) = 0.\end{aligned}$$