Generalizing Perfectness

Abstract. The paper deals about functions $f(n)$ and the question which numbers are $f(n)$-perfect. We achieved our solution with basic algebraic manipulations. At first we will denote some notations. All numbers which are $f(n)$ -perfect have a similar prime-factorization, where most times just the exponents of the primes are fixed. The first part of our solution considers $f(n) = \tau(n) + k$. We that just for $k \leq 0$ a number n can be $f(n)$ -perfect. In the next part we proof that only the powers of 2 are $\varphi(n)$-perfect. Then we will show, that all numbers $n = 2^k p$, where p is an prime number $p = 2^{k+1} + 2rk - 1$ are $f(n) = n + r$ -perfect. $f(n) = \ln(n)$ perfect are just numbers of the form p_1p_2 or third powers of prime numbers. For $f(n) = w_k^n$, we just get all solutions for $k = 2$ and 3. The last part of the problem deals with the case that $f_m(n)$ is an binomial coefficient. We found the necessary condition that $m - 1$ is just the product of different prime -numbers, for an $f_m(n)$ -perfect number.

Notations and some simplifications

Denote by

$$n = \prod_{i=1}^{N} p_i^{\alpha_i}$$

the prime-factorization of n. We can obviously calculate $\tau(n)$ in the following way, because we can choose the exponents $0, \cdots, \alpha_i$ for p_i independently.

$$\tau(n) = \prod_{i=1}^{N} (\alpha_i + 1)$$

To calculate $\varphi(n)$ we use the principle of inclusion and exclusion and so we get

$$\varphi(n) = n \cdot \prod_{i=1}^{N} \left(1 - \frac{1}{p_i}\right)$$

We have for each $n > 1 \in \mathbb{N}$:

$$f(n) = \sum_{1 \le d < n; d|n} f(d) \Leftrightarrow 2 \cdot f(n) = \sum_{1 \le d \le n; d|n} f(d)$$

Problem 1

a)

At first we will do some equivalent transformations:

$$f(n) = \tau(n)$$

$$2 \cdot \tau(n) = \sum_{1 \le d \le n; d|n} \tau(d) = \sum_{0 \le e_1 \le \alpha_1, \cdots, 0 \le e_N \le \alpha_N} \tau\left(\prod_{i=1}^{N} p_i^{e_i}\right)$$

$$2 \cdot \tau(n) = \sum_{0 \le e_1 \le \alpha_1, \cdots, 0 \le e_N \le \alpha_N} \prod_{i=1}^{N} (e_i + 1)$$

Obviously we can factorize this sum:

$$2 \cdot \tau(n) = \prod_{i=1}^{N} \sum_{e_i=0}^{\alpha_i} (e_i + 1) = \frac{\prod_{i=1}^{N} (\alpha_i + 1)(\alpha_i + 2)}{2^N}$$

$$2 \cdot \tau(n) = \frac{\tau(n) \cdot \prod_{i=1}^{N} (\alpha_i + 2)}{2^N}$$

Note that $\tau(n) > 0$. So we can divide this equality by $\tau(n)$.

$$2 = \prod_{i=1}^{N} \frac{\alpha_i + 2}{2}$$

With $\alpha_i \geq 1$ we get that

$$2 = \prod_{i=1}^{N} \frac{\alpha_i + 2}{2} \geq \left(\frac{3}{2}\right)^N$$

Now it obviously only holds for $N = 1$ and so we have $\alpha_1 + 2 = 4$, which implies that n is a square of a prime number.
Since we only used equivalent transformations n is $\tau(n)$-perfect iff n is a square of a prime number

b)

We generalize this function set $f(n) = \tau(n) + k$, where $k \in \mathbb{Z}$. It is trivial that $n = 1$ is never $\tau(n)+k$-perfect. So we can assume that $\tau(n) \geq 2$. Regarding the solution of Problem 1.a) we easily get:

$$2(\tau(n) + k) = \prod_{i=1}^{N} \frac{(\alpha_i + 1)(\alpha_i + 2)}{2} + k\tau(n)$$

Case 1: $k \geq 1$

$$2k \geq \tau(n) \left(\prod_{i=1}^{N} \frac{(\alpha_i + 2)}{2} + k - 2 \right) \geq \tau(n) \left(\left(\frac{3}{2}\right)^N + k - 2 \right)$$

Under the condition $k \geq 1$ the following equality holds:

$$\left(\frac{3}{2}\right)^N + k - 2 \geq 0$$

With $\tau(n) \geq 2$ we now get

$$k \geq \left(\frac{3}{2}\right)^N + k - 2 \Leftrightarrow 2 \geq \left(\frac{3}{2}\right)^N$$

So we get that $N = 1$.

$$2(\tau(p_1^{\alpha_1})+k) = \frac{(\alpha_1+1)(\alpha_1+2)}{2} + k\tau(p_1^{\alpha_1}) \Leftrightarrow 4((\alpha_1+1)+k) = (\alpha_1+1)(\alpha_1+2)+2k(\alpha_1+1)$$

$$2k(1-\alpha_1) = \alpha_1^2 - \alpha_1 - 2 = -(1-\alpha_1)(\alpha_1) - 2 \Leftrightarrow (\alpha_1-1)(\alpha_1+2k) = 2$$

But this equality has no solution, because $\alpha_1, k \geq 1$ is assumed.
Concluding, there is no $\tau(n) + k$ -perfect number for $k \geq 1$.

Case 2: $k \leq -1$

Our conjuncture is, that for each $k < 0$ there are infinity $f(n)$ -perfect numbers.

Problem 2

At first we will do some equivalent transformations:

$$f(n) = \varphi(n)$$

$$2\varphi(n) = \sum_{0 \leq e_1 \leq \alpha_1, \cdots, 0 \leq e_N \leq \alpha_N} \varphi\left(\prod_{i=1}^{N} p_i^{e_i}\right)$$

It is easy to ensure that $\varphi(mn) = \varphi(m)\varphi(n)$ if $(m,n) = 1$ (regarding to the explicit formula for $\varphi(n)$.)

$$_2\varphi(n) = \sum_{0 \leq e_1 \leq \alpha_1, \cdots, 0 \leq e_N \leq \alpha_N} \prod_{i=1}^{N} \varphi(p_i^{e_i})$$

We can obviously factorize this sum:

$$2\varphi(n) = \prod_{i=1}^{N} \sum_{e_i=0}^{\alpha_i} \varphi(p_i^{e_i})$$

It is very easy to ensure by induction over s that $\sum_{i=0}^{s} \varphi\left(p^i\right) = p^s$. Using this we get:

$$2\varphi(n) = \prod_{i=1}^{N} p_i^{\alpha_i} = n \Leftrightarrow \frac{1}{2} = \prod_{i=1}^{N} \left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{N} \left(\frac{p_i - 1}{p_i}\right)$$

Now we look at the biggest prime number dividing n, and like this it is very easy to see that the enumerator is never divisible by this prime number. But the value of the product has to be $\frac{1}{2}$. So we conclude that the biggest prime number is 2. Now we conclude that $n = 2^k$,with $k \geq 0$, is $\varphi(n)$-perfect. ($k = 0$ is pretty easy to verify.)

Problem 3

We will solve the problem for $f(n) = n + r$, where $r \in \mathbb{Z}$ Denote by p the natural number $2^{k+1} + 2kr - 1$, with an k such that p is an prime number. Now we will prove that $n = 2^k p$ is $f(n)$-perfect.

$$\sum_{1 \le d < n; d|n} f(d) = \sum_{1 \le d < n; d|n} (d + r) = \sum_{i=0}^{k-1} (2^i p + r) + \sum_{i=0}^{k} (2^i + r)$$

Using the formula for the sum of geometric progession we get:

$$\sum_{1 \le d < n; d|n} f(d) = \left(2^k - 1\right) p + kr + 2^{k+1} - 1 + (k+1)r - 1 + r$$

$$\sum_{1 \le d < n; d|n} f(d) = 2^k p - p + (2^{k+1} + 2kr - 1) + r = 2^k p + r = f(n)$$

Problem 4

Now we consider $f(n) = \ln(n)$ It is very easy to see that 1 is not $\ln(n)$-perfect. So we will assume that $n > 1$

$$2f(n) = \sum_{1 \le d \le n; d|n} f(d) = \sum_{1 \le d \le n; d|n} \ln(n) \Leftrightarrow n = \prod_{1 \le d < n; d|n} d$$

Now let n be $p_1^{\alpha_1} c$.

$$cp_1^{\alpha_1} = n \ge \prod_{i=0}^{\alpha_1 - 1} p_1^i c \ge c^{\alpha_1} \prod_{i=0}^{\alpha_1 - 1} p_1^i = c^{\alpha_1} p_1^{\frac{\alpha_1(\alpha_1 - 1)}{2}}$$

Looking at the prime factorization if n we get following conditions:

$$c \ge c^{\alpha_1} \Leftrightarrow c = 1 \wedge \alpha_1 = 1$$

For $c = 1$ we get

$$p_1^{\alpha_1} = p_1^{\frac{\alpha_1(\alpha_1 - 1)}{2}}$$

$$\alpha_1 = \frac{\alpha_1(\alpha_1 - 1)}{2} \Leftrightarrow 3 = \alpha_1$$

It can be easily verified that $n = p_1^3$ is $\ln(n)$-perfect.

For $c > 1$ we get $n = \prod_{i=1}^{N} p_i$. If $N > 3$ it is easy to see that there is more than one divisor of n which is divided by p_1. It is very easy to verify that $n = p_1 p_2$ is $\ln(n)$-perfect.

Problem 5

Denote by $w_t = \cos\left(\frac{2\pi}{t}\right) + i \cdot \sin\left(\frac{2\pi}{t}\right)$

$$f_t(n) = \omega_t^n$$

We were only able to solve the cases for $t = 2, 3$.

Case t=2

Case $p_1 = 2$**:** Let s be $\frac{n}{2^{\alpha_1}}$

$$f_t(n) = (-1)^n = 1 = \sum_{i=1}^{N} f_t(d) = \sum_{1\leq d<n; d|n} (-1)^d = \sum_{d|s; 1\leq d\leq s} (-1)^d + \sum_{2d|n; 1\leq 2d<n} (-1)^{2d}$$

$$1 = -\tau(s) + \sum_{d|\frac{n}{2}; 1\leq d<n} 1^d = -\tau(s) + \tau\left(\frac{n}{2}\right) - 1$$

$$2 = -\prod_{i=2}^{N}(\alpha_i + 1) + \alpha_1 \cdot \prod_{i=2}^{N}(\alpha_i + 1)$$

$$2 = (\alpha_1 - 1) \cdot \prod_{i=2}^{N}(\alpha_i + 1)$$

Now it is pretty easy to see that $n = 4 \cdot p_2$ or $n = 4 \cdot 2 = 8$

Case $p_1 > 2$**:**

$$(-1)^n = -1 = \sum_{1\leq d<n; d|n} (-1)^d = \sum_{1\leq d<n; d|n} -1 = -(\tau(n) - 1) \Leftrightarrow \tau(n) = 2$$

So n is an odd prime number.

Case t=3

Denote by

$$n = 3^r \cdot \prod_{i=1}^{N_\alpha} a_i^{\alpha_i} \cdot \prod_{i=1}^{N_\beta} b_i^{\beta_i}$$

Where a_i and b_i are the prime divisors of n holding $3|a_i - 1$ and $3|b_i - 2$.

Case $r = 0$: At first We will just look at the real parts of the numbers.

$$Re(w_3^n) = -\frac{1}{2} = \sum_{1 \leq d < n; d|n} Re(w_3^d) = -\frac{1}{2} \sum_{1 \leq d < n; d|n} 1$$

Now we can easily conclude that $\tau(n) = 2$. That means that n is a prime number p. Looking at the whole numbers we get: $w_3^p = w_3^1$.
So we have $n = p$ with $3|p-1$

Case $r \geq 1$: Let s be $\frac{n}{3^r}$. Looking at the real parts of the numbers we get:

$$Re(w_3^n) = 1 = \sum_{1 \leq d \leq s; d|s} -\frac{1}{2} + \sum_{1 \leq 3d < n; 3d|n} 1$$

$$1 = -\frac{\tau(s)}{2} + \sum_{1 \leq d < \frac{n}{3}; d|\frac{n}{3}} 1 = -\frac{\tau(s)}{2} + \tau\left(\frac{n}{3}\right) - 1$$

$$4 = -\prod_{i=1}^{N_\alpha}(\alpha_i + 1) \cdot \prod_{i=1}^{N_\beta}(\beta_i + 1) + 2r \cdot \prod_{i=1}^{N_\alpha}(\alpha_i + 1) \cdot \prod_{i=1}^{N_\beta}(\beta_i + 1)$$

$$4 = (2r - 1) \prod_{i=1}^{N_\alpha}(\alpha_i + 1) \cdot \prod_{i=1}^{N_\beta}(\beta_i + 1)$$

It is clear that $r = 1$, so there are these possibilities left for n: $3a_1^3, 3b_1^3, 3a_1b_1$. It is very easy to verify that all numbers of the form $3b_1^3$ or $= 3a_1b_1$ are w_3^n-perfect.

Problem 6

Define:

$$f_m(n) = \binom{m}{n}$$

Obviously no $n \leq m$ is $f_m(n)$ -perfect, because $\binom{m}{1} > 0$. Now we look at the prime factorization of $m - 1$:

$$m - 1 = \prod_{i=1}^{N} p_i^{\alpha_i}$$

Now we will show that if there is a k such that $\alpha_k > 1$, there will be no $f_m(n)$ -perfect number. Using Theorem 1 we get

$$p_k | f_m(n) = \binom{m}{n} = \sum_{1 \leq d < n; d|n} f_m(d) = \sum_{1 \leq d < n; d|n} \binom{m}{d}$$

$$p_k|\binom{m}{n} - \sum_{1<d<n;d|n} = \binom{m}{1} \Leftrightarrow p_k|\binom{m}{1}$$

This obviously is a contradiction.
Our conjuncture is that only $m-1$ is $f_m(n)$ -perfect iff $m-1$ is a prime number. To show that $m-1$ is $f_m(n)$-perfect if $m-1$ is a prime number is pretty easy, because

$$\binom{m}{m-1} = \binom{m}{1}$$

Theorem 1

$\alpha_k > 1$ and $1 \leq n \leq m$

$$p_k|\binom{m}{n} \Leftrightarrow 1 < n < m-1$$

Proof of Theorem 1

Proof. We use the explicit formula:

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

$$s = \sum_{i=1}^{\inf} \left\lfloor \frac{m}{p_k^i} \right\rfloor$$

It is clear that $p_k^s|m!$, but $p_k^{s+1} \nmid m!$ Similar we define

$$t = \sum_{i=1}^{\inf} \left\lfloor \frac{n}{p_k^i} \right\rfloor$$

$$u = \sum_{i=1}^{\inf} \left\lfloor \frac{m-n}{p_k^i} \right\rfloor$$

Looking at all singular summands we get:

$$\left\lfloor \frac{m}{p_k^i} \right\rfloor \geq \left\lfloor \frac{n}{p_k^i} \right\rfloor + \frac{m-n}{p_k^i} \right\rfloor$$

But for $i = \alpha_k$ and $m-1 > n > 1$ we have:

$$\left\lfloor \frac{m}{p_k^{\alpha_k}} \right\rfloor > \left\lfloor \frac{n}{p_k^{\alpha_k}} \right\rfloor + \left\lfloor \frac{m-n}{p_k^{\alpha_k}} \right\rfloor$$

So all in all we have:

$$s > t + u \Leftrightarrow p_k | \binom{m}{n}$$

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