Team France 3

8. **Discharged Batteries**

Abstract

In this problem, we have some charged and discharged batteries, and it is asked to find the best strategy possible to reduce the number of necessary tries to find a fixed number of charged batteries. It is a so-called "minmax" strategy, that is, the goal is to minimize the number tries in the worst possible case.

We treated the first two questions, that were only particular examples, and partly the questions 3 and 4. In both cases, we focused on c the number of charged batteries as a function of u the number of uncharged ones, giving upper bounds. We exhibited a "critic case" (corresponding to $c = u$ in the third question and $c = (k-1)u$ in the forth).

Notations

We define the following notations :
– let $b = c + u$ the number of batteries.
– let N the number of necessary n-tries.

n-tries when n batteries are necessary

In the third and forth question, we search asymptotic results. We take b such that $n \mid b$ to facilitate the manipulations, as our strategies often rely on making first a partition of the batteries in n-tries. In fact, we could increase b by putting news "fictive" uncharged batteries by the next multiple of n : these batteries are reprensented by randomly chosen batteries among the b batteries we have, and supposed uncharged : if they are charged, it only can increase the number of tries. We search asymptotic results, for b great compared to n (there are less than n fictive batteries). So we have only a small variation of b and N, so that this upper bound can be accurate. We remark that for $u < k$, N is increased by an affine function of u. Then, for $u = k$, there is a "critic case", and when u goes to b, N goes to a polynomial expression of u which is not linear anymore.

First case ("critic case") : nk batteries with $c = (n-1)k$ and $u = k$

Here we make k n-tries, doing a partition of the batteries. If all the n-tries are unsuccessful that means that each group of n batteries contains $(n-1)$ charged batteries and one uncharged battery. Now, we fix one battery in a group and make k n-tries with batteries of an other group of batteries. If the n-tries are once again unsuccessful it means that the battery we fix is uncharged, thus the $n-1$ other batteries of its group are charged. We take one battery of an other lot, if the new n-try is unsuccessful the following will be right because there is only one battery uncharged in each lot. Finally, in this case, we obtain the following upper bound :

$$N = k + n + 1$$

In the particular case of the third question, we have here $c = u$ and $N = k + 3$ (since $n = 2$). The first question is an even more particular case ($c = 5$).

Second case : $c > (n-1)k$

We use a similar strategy to solve this case.

The result will be he same but it will necessarly stay, after u tests, at least a pair of charged batteries. Thus, we have the following bound for $c > (n-1)u$:

$$N = u + k + 1.$$

Thus, we can match these two cases : if $c \geq (n-1)k$, we found $N = u + k + 1$.

Third case : $c < (n-1)k$

In this case, the preceding strategy is not so efficient, since there This time, it is not possible to increase $N(u)$ by a linear function of u. In fact, when c is as small as possible, that is $c = n$, there is only one try that can be successful. All other tries are unsuccessful and give only one information : at least one of the battery of the try is uncharged. Therefore, in the worst case, one has to make all possible tries but one (the successful one). Thus, we obtain :

$$N = \binom{b}{n} - 1 = \binom{n+u}{n} - 1$$

which is a polynom in u of degree n.

Particular case : $n = 3$

We treat here the second question, with $b = 8$. This particular case corresponds to the third case : $c < (n-1)k$. Failing to have found a complete strategy to minimize the number of 3-tries we found an increase by introducing an other "fictive" uncharged battery so we have $c = 4$ and $u = 5$ that means $b = 9$ and we can do three 3-tries. If the three 3-tries are unsuccessful, that means that there is 2 lots with 2 uncharged batteries and 1 charged battery, the last lot will be constitued with 2 charged batteries and 1 uncharged batteries. So we take two lots and we can make $\binom{6}{3} = 20$ combinations .
If the 20 3-tries are unsuccessful that means that we took the lots with one charged batteries so we juste have to take a lot with one uncharged battery and take the other lot with two charged batteries and do 19 3-tries because the last cambination will be successful. Thus by introducing another uncharged batteries we know that the number of 3-tries with 8 batteries equally distributed is increased by

$$N = 42$$

Further ideas

We decided here to determinate a strategy to minimize the number of tries in the worst case, but one could also try for example to have a strategy in order to reduce the average number of tries, which would lead to study probabilities.