Probleme 4: Isosceles Triangles.

Team France 3

Abstract

The solution of this problem has multiple origins. Some questions were answered by France 2 team during the French tournament. We could not find a better solution for several questions and we used those results in order to answer some other questions, so we retrace them here.

For the first question we have found different algebraic and geometric solutions. The easiest one is the last one.

The solution for the second question was found by France 2 team. This solution is very algebraic, using sine law in different triangles, Steward's theorem and a special case of Steiner's theorem. We judged unnecessary to show proofs of inequalities, as it is quite obvious. The answer to que question 3 is an impressif geometric counterexample found on internet by France 2 team.

The solutions for questions 4 and 5 are of the same style as question 2, found by France 2 team.

In the question 6 we present a counterexample that we have found using GeoGebra, for a negative value of n. Nevertheless, the proof itself doesn't use GeoGebra and is strictly rigourous.

We found the question 7 quite weird, because previous questions already gave the whole answer.

In the part 8 we asked ourselves (in order to enlarge previous two questions), for which values of n, are those affirmations true? Nevertheless, we haven't got time to find a strict answer. We also give an exemple for a particular case where we can give a partial empiric answer.

Important remark

Let ABC be a triangle isosceles in A. Then the angle bisector of $\widehat{A}$ is also the median and the height of ABC and thus the axis of symetry of the triangle. Then obviously all similar objects, attached to B and C (internal or external angle bisectors, medians, n-lines...) are symetric and thus all the corresponding segments have the same length. That means that in all the questions of this problem we need only answer the question "Is it true that if the segments are of the same length then the triangle is isosceles?"

1 Internal angle bisectors

As our team comes from different places there are several solutions of different nature.

Algebraic solution

Let ABC be a triangle and $[CK]$ and $[BJ]$ the internal bisectors of $\widehat{C}$ and $\widehat{B}$ respectively.

Let I be the intersection point of $[CK]$ and $[BJ]$. Then $I = \text{bar}\{A(a), B(b), C(c)\}$ as $K = \text{bar}\{A(a), B(b)\}$. Thus:

$$BJ^2 = \frac{(a\overrightarrow{BA} + c\overrightarrow{BC})^2}{(a+c)^2} = \frac{1}{(a+c)^2} \times (a^2c^2 + c^2a^2 + 2ac\overrightarrow{BA} \cdot \overrightarrow{BC}) = \frac{2a^2c^2}{(a+c)^2}(1 + \cos \widehat{B})$$

and

$$CK^2 = \frac{2a^2b^2}{(a+b)^2}(1+\cos\widehat{C}).$$

If the length a and the measure of the angle $\widehat{A}$ are fixed and b and c vary, then A runs over an arc of a circle subtended by the chord $[BC]$.

Suppose $c > b$. Then $\cos\widehat{B} > \cos\widehat{C}$ et $\dfrac{ac}{a+c} > \dfrac{ab}{a+b}$, and we have

$$2(\frac{ac}{a+c})^2(1+\cos\widehat{B}) > \frac{2a^2b^2}{(a+b)^2}(1+\cos\widehat{C}).$$

In the same way, if $c < b$ then

$$2(\frac{ac}{a+c})^2(1+\cos\widehat{B}) > \frac{2a^2b^2}{(a+b)^2}(1+\cos\widehat{C}).$$

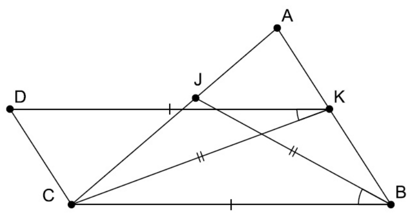
Thus $BJ = CK$ if and only if $b = c$, i.e. if and only if ΔABC is isosceles.

First geometric solution.

First we need the following lemma.

Lemma 1. *A quadrangle that has two opposite sides of the same length and two opposite obtuse angles of the same measure is a parallelogram.*

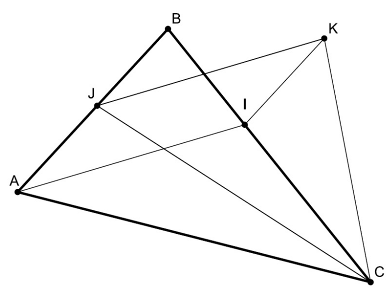
Using this lemma we can now prove the desired result.



Let the point D be such that $DJ = BC$ and $\widehat{DJB} = \widehat{KCB} = \gamma$ as on the picture. Let's note $\beta = \widehat{JBC}$. In the triangles BJD and BKC we have $BJ = KC$, $JD = CB$ and $\widehat{DJB} = \widehat{KCB} = \gamma$. Then the triangles are isometric. Thus $\widehat{JDB} = \widehat{CBK} = 2\beta$, and $\widehat{DJC} = \widehat{DBC} = \pi - \beta - \gamma > \pi/2$ as $\beta + \gamma < \pi/2$.

By the lemma above, $BCJD$ is a parallelogram, thus $\beta = \gamma$ and we have $\widehat{B} = \widehat{C}$ i.e. the triange ABC is isocele.

Second geometric solution.



Suppose that ABC is a non-isosceles triangle, say $AB > AC$. Let CI and BJ be its internal angle bisectors that we soppose to be of the same length. Let K be a point such that $CIKJ$ is a parallelogram. Then $\widehat{BCI} = \widehat{ICA} = \widehat{JKI}$ and $\widehat{ABJ} = \widehat{JBC}$. It is clear ($AB > AC$) that $\widehat{BCI} > \widehat{JBC}$. As $CI = BJ$, the triangle BJK is isosceles in J, namely $\widehat{JKB} = \widehat{JBK}$. Thus, $\widehat{BCI} + \widehat{IKB} = \widehat{JBC} + \widehat{KBI}$. Then $\widehat{IKB} < \widehat{KBI}$.

On the other hand, by the cosine law in triangles BCJ and CIB, as $CB = CB$, $BJ = CI$ and $\widehat{BCI} > \widehat{JBC}$, we have $CJ < BI$. Hence, $KI < BI$. Hence, $\widehat{IKB} > \widehat{KBI}$. That leads to a contradiction. Thus "if a triangle is non-isosceles, none of his internal angle bisectors are equal".

The solutions in the following four sections were found by Team Orsay at French Tournament. Wee coud not find a better solution and we need their results for the last questions.

2 Symmedians

We will use standard notations: $BC = a$, $CA = b$ and $AB = c$.

We need two following results:

Theorem 1 (Steward's theorem)**.** *In a triangle ABC, with cevian AA', we have the relation:*

$$BC * (AA'^2 + BA' * CA') = AC^2 * BA' + AB^2 * CA'$$

Lemma 2 (Special case of Steiner theorem)**.** *Let ABC be a triangle and BH (respectively BJ) its median (respectively symmedian) at B. Then*

$$\frac{AJ}{c^2} = \frac{CJ}{a^2} = \frac{b}{a^2+c^2}.$$

PROOF. By definition of the symmedian, $\widehat{CBJ} = \widehat{ABH}$. Let's note x the mesure of $\widehat{ABH}$ and y the measure of $\widehat{HBJ}$. Then, by the sine law in triangles ABH and CBH,

$$\frac{AH}{sin(x)} = \frac{BH}{sin(\widehat{A})} \text{ and } \frac{CH}{sin(x+y)} = \frac{BH}{sin(\widehat{C})}.$$

As $AH = CH$, the sine law in the triangle ABC combined with two previous equalities yield

$$\frac{sin(x+y)}{sin(x)} = \frac{c}{a}.$$

Next, the sine law in triangles ABJ and CBJ gives

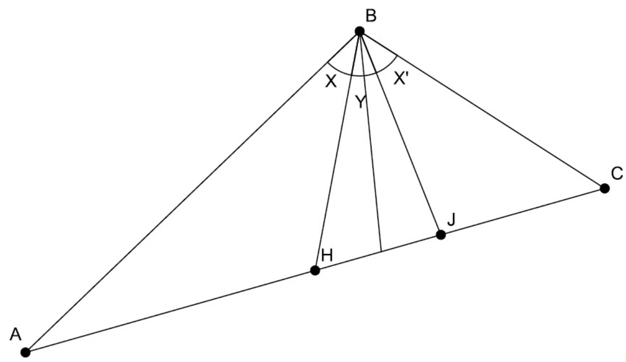
$$\frac{AJ}{sin(x+y)} = \frac{BJ}{sin(\widehat{A})} \text{ and } \frac{JC}{sin(x)} = \frac{JC}{sin(x)}.$$

Thus,

$$\frac{AJ}{JC} = \frac{sin(x+y)}{sin(x)} \cdot \frac{c}{a} = \frac{c^2}{a^2}.$$

that is equivalent to the desired equality.

Let $[BH]$ (resp. $[BJ]$) be the median (resp. symmedian) of ABC at B.



Using Steward's theorem in triangle ABC with cevian BJ and the previous lemma, we obtain:

$$b(BJ^2 + AJ * JC) = a^2 * AJ + c^2 * CJ,$$

thus

$$BJ^2 = \frac{2a^2c^2}{a^2+c^2} - \frac{a^2b^2c^2}{(a^2+c^2)^2}.$$

Analogically,

$$CI^2 = \frac{2a^2b^2}{a^2+b^2} - \frac{a^2b^2c^2}{(a^2+b^2)^2}.$$

If $b \neq c$, without any loss of generality we may suppose that $c > b$. If $c > b$, it is obvious that

$$\frac{2a^2c^2}{a^2+c^2} > \frac{2a^2b^2}{a^2+b^2}$$

and

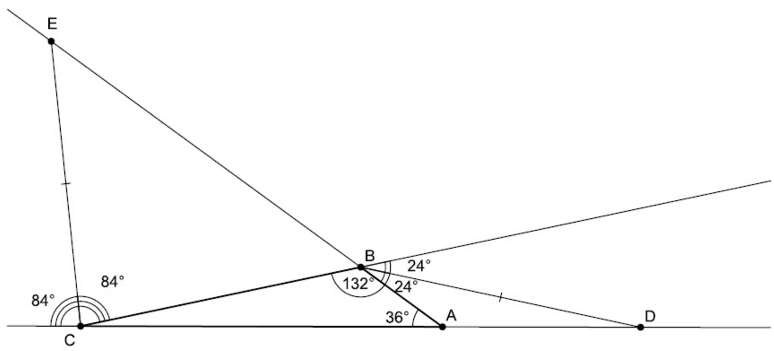
$$-\frac{a^2b^2c^2}{(a^2+c^2)^2} > -\frac{a^2b^2c^2}{(a^2+b^2)^2}.$$

Thus if $c > b$, then $BJ^2 > CI^2$.

That means that if $BJ = CI$ then $c = b$.

3 External angle bisectors

The following counterexemple was found in the literature by Team Orsay, but we don't have the exact reference.



On the figure above, the triangle ABC is defined by his angles: $\widehat{A} = 36$, $\widehat{B} = 132$, thus $\widehat{C} = 12$. CE is the external bisector at C. Thus, $\widehat{ECB} = \frac{\pi - \widehat{C}}{2} = 84$. $\widehat{CBE} = \pi - \widehat{B} = 48$, then $\widehat{CEB} = \pi - \widehat{ECB} - \widehat{CBE} = 48$. Thus, $CE = CB$.

Next, let BD be the external bisector at B. Thus, $\widehat{ABD} = \frac{\pi - \widehat{B}}{2} = 24$. $\widehat{BAD} = \pi - \widehat{A} = 144$, then $\widehat{BDA} = \pi - \widehat{ABD} - \widehat{BAD} = 12$. Thus, $CB = BD$.

Thus, $CE = BD$. So, here we have an example of a non-isosceles triangle with two equal external bisectors.

4 Exsymmedians

We will suppose that the triangle ABC is non-isosceles. Let AA' be the external bisector at A and AA'' be the ex-symmedian. Without loss of generality, suppose that $b > c$, so that B is between A'' and C. We also have

$$\widehat{CA'A} = \pi - (\widehat{C} + \frac{\pi + \widehat{A}}{2}) = \widehat{B} - \frac{\pi - \widehat{A}}{2}$$

By definition of ex-symmedians, we obtain

$$\widehat{A''AB} = \frac{\pi - \widehat{A}}{2} - \widehat{A'AA''} = \widehat{C}.$$

Using sine law in triangles $A''AB$ and $A''AC$,

$$\frac{A''C}{sin(\hat{A}+\hat{C})} = \frac{AA''}{sin(C)},$$

$$\frac{A''B}{sin(\hat{C})} = \frac{A''A}{sin(\pi-\hat{B})}.$$

Using the sine law in triangle ABC, we obtain

$$\frac{A''C}{A''B} = \frac{\sin^2(\hat{B})}{\sin^2(\hat{C})} = \frac{b^2}{c^2}.$$

The same result can be obtained when $b < c$.

Lemma 3. *The lenght of the ex-symmedian at A is* $\frac{abc}{|b^2-c^2|}$.

PROOF. Without loss of generality, suppose that $b > c$. Using that $\frac{A''C}{A''B} = \frac{b^2}{c^2}$ and $A''C = A''B + a$, we obtain $A''C = \frac{ab^2}{b^2-c^2}$ and $A''B = \frac{ac^2}{b^2-c^2}$. Then, using Stewart's theorem in triangle $AA''C$ with cevian AB, we obtain:

$$c^2A''C + aA''C * A''B = a * AA''^2 + A''B * b^2.$$

As $c^2 * A''C = b^2 * A''B$, we obtain

$$AA'' = \sqrt{A''C * A''B} = \frac{abc}{b^2-c^2},$$

finishing the proof.

It is now clear that it is enough to ensure $a^2 - b^2 = b^2 - c^2$, i.e. $a^2 + c^2 = 2b^2$, in order to construct non-isosceles triangles having two equal ex-symmedians.

5 1-lines and 2-lines

By the angle bisector theorem, it is obvious that internal angle bisectors are internal 1-lines. The fact that symmedians are internal 2-lines was solved in question 2 (cf. lemma 2)

Let AA' be the external bisector at A. It is clear that A' is not on the segment BC. Without loss of generality, suppose that B is between A' and C. Using the sine law in triangles $AA'B$ and $AA'C$, we obtain:

$$\frac{BA'}{\sin\frac{\pi-\widehat{A}}{2}} = \frac{AA'}{\sin\widehat{B}}$$

and

$$\frac{CA'}{\sin\frac{\pi+\widehat{A}}{2}} = \frac{AA'}{\sin\widehat{C}}.$$

Comparing the two relations and using $\frac{\sin\widehat{B}}{\sin\widehat{C}} = \frac{b}{c}$ and $\sin\frac{\pi-\widehat{A}}{2} = \sin\frac{\pi+\widehat{A}}{2}$ yields $\frac{BA'}{CA'} = \frac{c}{b}$, proving that external bisectors are external 1-lines.

The fact that the ex-symmedians are external 2-lines was proved in question 4.

6 Internal n-lines

Preliminary result

Let AA_n be an internal n-line. Using Stewart's theorem in triangle ABC with cevian AA_n we obtain

$$a(AA_n^2 + BA_n \cdot CA_n) = b^2 \cdot BA_n + c^2 \cdot CA_n$$

so that using the definition of internal n-line we obtain:

$$BA_n = \frac{ac^n}{c^n + b^n}$$

$$AA_n^2 = \frac{b^2c^n + c^2b^n}{b^n + c^n} - \frac{a^2b^nc^n}{(b^n + c^n)^2}$$

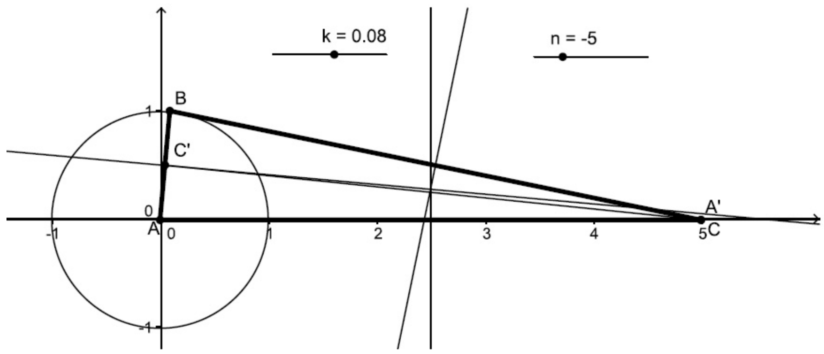
The answer

The question was: Is it true that two internal n-lines of a triangle are equal if and only if the triangle is isosceles? The answer to this question is "no", that may be shown by the following counterexemple, constructed using GeoGebra.

Let ABC be a triangle such that $AC = 5$ and $AB = 1$. It is clear (triangular inequality) that BC cannot be shorter than 4 nor longer than 6. So, BC is between 4 and 6.

Fix $n = -5$.

On the figure below, C_{-5} is called C' and A_{-5} is called A'.



Take B on the circle of centre O and of radius 1, such that the ordinate of B is positive. Let's call k its abscissa. In that case, its ordinate is $\sqrt{1-k^2}$.

Let's prove that the variation of lengths AA' and CC' is a continuous function of k. It is clear that the coordinates of B are continuous functions of k. Thus, the length BC is continuous function of k. And, as $AA_n^2 = \frac{b^2c^n + c^2b^n}{b^n + c^n} - \frac{a^2b^nc^n}{(b^n + c^n)^2}$ (see the previous question), that means that the lengths of AA' and CC' are also continuous functions of BC, thus continuous functions of k.

The triangle ABC cannot be isosceles in A nor in B, however it can be isosceles in C. In the latter case, $AC = 5$, $AB = 1$ and $BC = 5$. Let's calculate AA' and CC':

$$AA'^2 = \frac{5^2 * 1^{-5} + 1^2 * 5^{-5}}{5^{-5} + 1^{-5}} - \frac{5^2 * 1^{-5} * 5^{-5}}{(5^{-5} + 1^{-5})^2} \approx 24,984,$$

that's why on the figure above A' coinsides with C, and

$$CC'^2 = \frac{5^2 * 5^{-5} + 5^2 * 5^{-5}}{5^{-5} + 5^{-5}} - \frac{1^2 * 5^{-5} * 5^{-5}}{(5^{-5} + 5^{-5})^2} \approx 24,750.$$

So, if ABC is isosceles in C, $AA' \neq CC'$.

Using the same GeoGebra model we may vary the value of k.

Let's take $k = 1$. In that case, $BC = 4$ and we obtain

$$AA'^2 = \frac{5^2 * 1^{-5} + 1^2 * 5^{-5}}{5^{-5} + 1^{-5}} - \frac{4^2 * 1^{-5} * 5^{-5}}{(5^{-5} + 1^{-5})^2} \approx 24,987$$

and

$$CC'^2 = \frac{5^2 * 4^{-5} + 4^2 * 5^{-5}}{4^{-5} + 5^{-5}} - \frac{1^2 * 4^{-5} * 5^{-5}}{(4^{-5} + 5^{-5})^2} \approx 22,593.$$

So, if $k = 1$, $AA' > CC'$.

Now, let's take $k = -1$. In that case, $BC = 6$ and we obtain

$$AA'^2 = \frac{5^2 * 1^{-5} + 1^2 * 5^{-5}}{5^{-5} + 1^{-5}} - \frac{6^2 * 1^{-}5 * 5^{-5}}{(5^{-5} + 1^{-5})^2} \approx 24,981$$

and

$$CC'^2 = \frac{5^2 * 6^{-5} + 6^2 * 5^{-5}}{6^{-5} + 5^{-5}} - \frac{1^2 * 6^{-5} * 5^{-5}}{(6^{-5} + 5^{-5})^2} \approx 32,642.$$

So, if $k = -1$, $AA' < CC'$.

But the variation of AA' and CC', and also of $AA' - CC'$ is a continuous function of k, thus we can deduce that there exists at least one value of k between -1 and 1 that we will call k', such that $AA' - CC' = 0$, thus $AA' = CC'$. And, we know that for k', ABC is not isosceles, because as we have seen for the only case when ABC is isosceles, $AA' \neq CC'$.

That means that for $k = k'$, $AA' = CC'$ and ABC is not isosceles.

7 External n-lines

The external bisectors are external 1-lines, as shown in the question 5. Thus the counterexemple un the question 3 shows that the equality of external n-lines does not, in general, imply that the triangle is isosceles.

8 Additional researche

In questions 6 and 7 we proved that the affirmations do not work in general, but we may ask the following question, "For which values of n are these affirmations true?"

Question 6

It seems that the affirmation of question 6 is true for all positive values of n. To prove it, it is enough to prove that for all positive n, if $a > b$, then $AA_n < BB_n$.

To prove this, it is enough to prove that, when $n > 0$, $a > b > 0$ and $c > 0$,

$$\frac{b^2c^n + c^2b^n}{b^n + c^n} - \frac{a^2b^nc^n}{(b^n + c^n)^2} < \frac{a^2c^n + c^2a^n}{a^n + c^n} - \frac{b^2a^nc^n}{(a^n + c^n)^2}$$

For the moment, we have no idea how to prove this inequality, but it seems to work, as we tried it for different values of a, b, c.

8.1 Other directions

We could also try to find if there a some values of n, such that the affirmation of question 7 does work, or prove that this affirmation is false for all possible values of n.

We could also find the exact value of k' for question 6. Geogebra tells, $k' \approx 0.06$. Other contreexamples could be found (but, we haven't really found an answer to these questions)

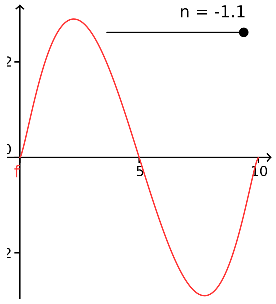
8.2 An interesting case for question 6

For the case of the triangle ABC where B is on segment AC, by a geogebra simulation, we found that strangely the only values of n, for which the affirmation of question 6 is not correct are between -1.1 and -1.4.

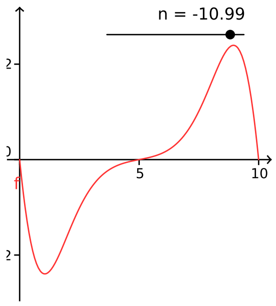
In fact, we fixed $AC = 10$, and for each value of n, we asked geogebra to draw a graphic showing the difference between lenghts of internal n-lines from A and from C in function of the lenght AB. (The lenght of the internal n-line from B seems to be always the lowest.)

Usually the only case when the graphic showed that this difference was equal to 0 was when $AB = 5$, $AB = 0$ or $AB = 10$, i.e. when the triangle is isosceles. That showed that in most cases this sort of triangles respect the affirmation of question 6.

For example, for $n = -1.1$:

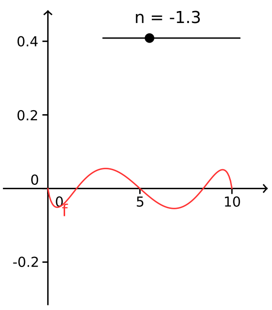


Or, for $n = -10.99$:

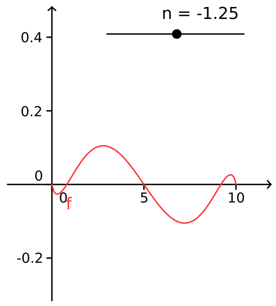


But, for some special values of n, more precisely between -1.1 and -1.4, the difference is equal to 0 for more values of AB than before.

For example, for $n = -1.3$:



Or, for $n = -1.25$:



Interesting remark

Let's call x the lenght AB. What is interesting for that sort of triangles is that the expression of lenghts AA_n and CC_n is more easy. In fact,

$$AA_n = \frac{(10-x)x^n}{x^n + 10^n} + x = \frac{(10-x)}{1 + \frac{10}{x}^n} + x$$

and

$$CC_n = \frac{x(10-x)^n}{(10-x)^n + 10^n} + 10 - x = \frac{x}{1 + \frac{10}{10-x}^n} + 10 - x.$$

Suppose, without any loss of generality, that $x > 10 - x$, i.e $c > a$ (using standart notations for a triangle).

In that case, as

$$\frac{10}{x} < \frac{10}{10-x},$$

we have

$$\frac{(10-x)}{1 + \frac{10}{x}^n} > \frac{x}{1 + \frac{10}{10-x}^n}$$

and thus $AA_n > CC_n$.

So, we have proved that for those triangles the affirmation of question 6 is true for all positive n.