Problem 10: Densities of Natural Subsets.

Team France 3

Abstract

In this problem, we try to find a way to "count" the number of elements in a subset of $\mathbb{N}$, especially infinite subsets. In order to do that, we will study some " densities ", which can indicate the number of elements in a subset, comparing to the number of elements in $\mathbb{N}$.

This notion of density is really intuitive : it seems evident that there is more even natural than number which can be divided by 3. But the sets of this numbers can both be put in bijection with $\mathbb{N}$; the notion of density help us to solve this problem. We answer to questions 1,2,3,4 and 6. We have a conjecture for question 7, and we treated some cases for questions 5 and 8.

Preliminary remarks

We say that a property is true for μ if it is true for μ_1, μ_2, and μ_3.

We first obtain easily the following property:
If A and B are two subsets of $\mathbb{N}$ such that $A \subset B$, then $\mu(A) \leq \mu(B)$. In fact, for $\mu(2)$ and $\mu(3)$, for x close enough to 1, the terms of the sums are positive, and concerning $\mu(1)$, for all positive integer n, we have:

$$\frac{\#(A \cap [1,n])}{n} \leq \frac{\#(B \cap [1,n])}{n}$$

and the conclusion follows.

By definition

$$\limsup_{n\to+\infty} u_n = \lim_{n\to+\infty} \sup\{(u_m)_{m\geq n}\},$$

so if $(u_n)_{n\in\mathbb{N}}$ converges to a finite limit then its sub-sequence $(\sup\{(u_m)_{m\geq n}\})_{m\in\mathbb{N}}$ converges to the same limit and we have

$$\limsup_{n\to+\infty} u_n = \lim_{n\to+\infty} u_n.$$

The similar statement is true for the functions admitting a finite limit.

Some lemmas

Lemma 1

$$\sum_{n\in\mathbb{N}} \frac{1}{n^x} \underset{1^+}{\sim} \frac{1}{x-1}.$$

Proof : First, for any integer $n > 1$, we have

$$\int_n^{n+1} \frac{1}{t^x} dt \leqslant \frac{1}{n^x} \leqslant \int_{n-1}^{n} \frac{1}{t^x} dt.$$

And so, by summing those inequalities over n, we find:

$$\int_1^{\infty} \frac{1}{t^x} dt \leqslant \frac{1}{n^x} \leqslant \int_0^{\infty} \frac{1}{t^x} dt,$$

that gives us

$$\frac{1}{x-1} \leqslant \sum_{n\in\mathbb{N}} \frac{1}{n^x} \leqslant 1 + \frac{1}{x-1}.$$

Thus

$$1 \leq (x-1) \sum_{n\in\mathbb{N}} \frac{1}{n^x} \leq 1 + x - 1,$$

that yelds, taking the limit at $x \to 1^+$, the desired equivalence

$$\sum_{n\in\mathbb{N}} \frac{1}{n^x} \underset{1^+}{\sim} \frac{1}{x-1}. \qquad \square$$

Lemma 2 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences such that $u_n \leqslant v_n$. Then, if $U = \{u_k, k \in \mathbb{N}\}$ and $V = \{v_k, k \in \mathbb{N}\}$, we have $\mu(V) \leq \mu(U)$.*

Proof : For μ_1 : it is clear that in $[|1,n|]$ there is at least as many elements of u_n as those of v_n. So $\#(U \cap [|1,n|]) \geqslant \#(V \cap [|1,n|])$. Dividing by n and taking the limit, we have $\mu_1(V) \leqslant \mu_1(U)$.

For μ_2 : Introducing $U_n(x)$ and $V_n(x)$ two functions defined on $[0;1]$ by $U_n(x) = (1-x)\sum_{k\in U\cap[|1,n|]} x^k$ (and similarly for $V_n(x)$), we have for the same reason $U_n(x) \leqslant V_n(x)$ for all $x \in [0;1]$.

Thus, takind the limit whent n tends to ∞, and noting $U(x) = U_\infty(x)$ (which does converge near 1^-, see question 1), we have $U(x) \leq V(x)$. Finally, when x tends to 1^- we have : $\mu_2(V) \leqslant \mu_2(U)$.

By a similar reasoning, we have the same result for μ_3 $\square$.

Question 1

We compute $\mu(\mathbb{N})$:

We have easily for $\mu_1(\mathbb{N}) = 1$:

$$\mu_1(\mathbb{N}) = \limsup_{n\to+\infty} \frac{\#(\mathbb{N} \cap [1,n])}{n} = \limsup_{n\to+\infty} \frac{\#([\![1,n]\!])}{n} = \limsup_{n\to+\infty} \frac{n}{n} = 1.$$

For μ_2, as $x \to 1^-$ we may suppose $x \in [0;1[$. Then we have

$$(1-x)\sum_{n\in\mathbb{N}} x^n = (1-x) \times \frac{1}{1-x} = 1$$

and thus

$$\mu_2(\mathbb{N}) = \lim_{x\to 1^-} 1 = 1.$$

For μ_3, since $\displaystyle\sum_{n\in\mathbb{N}} \frac{1}{n^x} \underset{1^+}{\sim} \frac{1}{x-1}$ by the Lemma 1 we have:

$$\lim_{x\to 1^+} (x-1) \times \sum_{n\in\mathbb{N}} \frac{1}{n^x} = (x-1) \times \frac{1}{x-1} = 1.$$

Thus, $\limsup\limits_{x\to 1^+} (x-1) \sum_{n\in\mathbb{N}} \frac{1}{n^x} = \lim\limits_{x\to 1^+} (x-1) \times \sum_{n\in\mathbb{N}} \frac{1}{n^x}$, and

$$\mu_3(\mathbb{N}) = 1.$$

Furthermore, it is clear that μ is positive, so we have for all subset A of $\mathbb{N}$:

$$0 \leq \mu(A) \leq 1.$$

And so μ is well-defined, since the superior limit of a bounded function or sequence always exists.

Question 2

Let E be a finite subset, and p its cardinal. We have:

$$\mu_1(E) = \limsup_{n\to+\infty} \frac{\#(E \cap [1,n])}{n}.$$

For $n > max(E)$, we have $\dfrac{\#(E \cap [1,n])}{n} = \dfrac{p}{n}$, thus

$$\mu_1(E) = \limsup_{n\to+\infty} \frac{p}{n}.$$

The limsup of a converging sequence is its limit, hence :

$$\mu_1(E) = 0.$$

Let now $P(x) = (1-x) \times \sum_{n \in E} x^n = p$ for any real x, P is a polynomial (since E is finite). Thus it is a continous function such that $P(1) = 0$, therefore:

$$\mu_2(E) = \limsup_{x \to 1^-} P(x) = \lim_{x \to 1^-} P(x) = 0.$$

For μ_2, we consider the function Q defined by $Q(x) = (x-1) \times \sum_{n \in E} \frac{1}{n^x}$. For $x \geq 1$, as E is finite, Q is continous, and $Q(1) = 0$, so:

$$\mu_3(E) = \limsup_{x \to 1^-} Q(x) = \lim_{x \to 1^-} Q(x) = 0.$$

Thus, we obtain:

$$\mu(E) = 0.$$

Question 3

Let F such that $F = \{a + kd / k \in \mathbb{N}\}$, where a and d are two non negative integers. We give here values of $\mu(F)$.

$$\mu_1(F) = \limsup_{n \to +\infty} \frac{\#((a + kd, k \in \mathbb{N}) \cap [1, n])}{n}.$$

We have for $n \geq d$: $\#((a + kd, k \in \mathbb{N}) \cap [1, n]) = 1 + \lfloor \frac{n-a}{d} \rfloor$.
We have $1 + \lfloor \frac{n-a}{d} \rfloor \underset{+\infty}{\sim} \frac{n}{d}$, therefore:

$$\mu_1(F) = \frac{1}{d}.$$

In order to study $\mu_2(F)$, let's first look at $(1-x) \sum_{n \in F} x^n$:

$$(1-x) \sum_{n \in E} x^n = (1-x) \sum_{k \in \mathbb{N}} x^{a+kd} = (1-x) x^a \sum_{k \in \mathbb{N}} (x^d)^k.$$

We may suppose $x \in [0; 1[$, so

$$(1-x) x^a \sum_{k \in \mathbb{N}} (x^d)^k = (1-x) x^a \lim_{k \to +\infty} \frac{(x^d)^{k+1} - 1}{x^d - 1} - 1$$
$$= (1-x) x^a \left(\frac{1}{1 - x^d} - 1\right) = x^a \frac{1}{\sum_{i=0}^{d-1} x^i} + x^{a+1} - 1.$$

With f such that $f(x) = x^a \frac{1}{\sum_{i=0}^{d-1} x^i} + x^{a+1} - 1$ for x close to 1, we have f continous and $f(1) = \frac{1}{d}$, so :

$$\mu_2(E) = \limsup_{x \to 1^+} f(x) = \lim_{x \to 1^+} f(x) = \frac{1}{d}.$$

In the same way for μ_2 we haveb

$$\mu_3(F) = \limsup_{x\to 1^+}(x-1)\sum_{n\in E}\frac{1}{n^x}$$

$$\mu_3(F) = \limsup_{x\to 1^+}(x-1)\sum_{k\in\mathbb{N}}\frac{1}{(a+kd)^x}$$

$$\mu_3(F) = \limsup_{x\to 1^+}(x-1)\sum_{k\in\mathbb{N}}\frac{1}{(\frac{a}{d}+k)^x d^x}$$

$$\mu_3(F) = \limsup_{x\to 1^+}\frac{(x-1)}{d^x}\sum_{k\in\mathbb{N}}\frac{1}{(\frac{a}{d}+k)^x}$$

And we have $\sum_{k\in\mathbb{N}}\frac{1}{(\frac{a}{d}+k)^x}\underset{1^+}{\sim}\frac{1}{x-1}$ because :

$$\sum_{k\in\mathbb{N}}\frac{1}{(\lfloor\frac{a}{d}\rfloor+k)^x}\leqslant\sum_{k\in\mathbb{N}}\frac{1}{(\frac{a}{d}+k)^x}\leqslant\sum_{k\in\mathbb{N}}\frac{1}{(\lfloor\frac{a}{d}\rfloor+1+k)^x}$$

and both sides of the inequality are equivalent to $\frac{1}{x-1}$ to 1. Thus :

$$\mu_3(F) = \limsup_{x\to 1^+}\times\frac{1}{d^x}\frac{x-1}{x-1} = \frac{1}{d}.$$

So for any μ, we have :

$$\mu(F) = \frac{1}{d}$$

Question 4

The formula in question 6 is true for μ when B is finite (with p its cardinality). Indeed, we want to prove $\mu(A\cup B) = \mu(A)$ for all natural subset A. And we have : For μ_1 :

$$\begin{aligned}\mu_1(A\cup B) &= \limsup_{n\to+\infty}\frac{\#((A\cup B)\cap[1,n])}{n} = \limsup_{n\to+\infty}\frac{\#(A\cap[1,n])}{n}+\frac{p}{n}\\ &= \limsup_{n\to+\infty}\frac{\#(A\cap[1,n])}{n} = \mu_1(A).\end{aligned}$$

For μ_2 :

$$\begin{aligned}\mu_2(A\cup B) &= \limsup_{x\to 1^-}(1-x)\sum_{n\in A\cup B}x^n\mu_2(A\cup B)\\ &= \limsup_{x\to 1^-}(1-x)(\sum_{n\in A}x^n+\sum_{n\in B}x^n-\sum_{n\in A\cap B}x^n).\end{aligned}$$

And, when x is close enough to 1, $(1-x)\sum_{n\in B} x^n$ and $(1-x)\sum_{n\in A\cap B} x^n$ are close enough to 0. So :

$$\mu_2(A\cup B) = \mu_2(A).$$

And we can have the similar proof for μ_3.

Let x a real greater than 1 (if $x=1$ then $E=\{1\}$ and thus $\mu(E)=0$). We make two remarks : first, the formula of question 6 is true for μ when B is finite. Second, $\exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$, and we have $v_n = \lfloor x^n \rfloor \geqslant d\times n = u_n$, for any chosen d. And, by the first remark, we can ignore the terms before n_0 (as there are a finite number of them). So, using the second lemma, and with the same notations, we have : $\mu(V) \leqslant \frac{1}{d}$ for all $d\in\mathbb{N}$. Finally, $\mu(V)=0$.

Question 5

Let (u_n) be a positive recurrent sequence of order 2 defined by $\forall n\in\mathbb{N}, u_{n+2} = au_{n+1}+bu_n$ where a and b ar ereal numbers, and u_1 and u_2 are given (and $(u_n)>0$). We try to find the density of $U=\{u_n, n\in\mathbb{N}\}$. First, we know that find a formula for (u_n) is equivalent to solve the following equation of order 2,

$$(E)\,:\, x^2 - ax - b = 0.$$

We call $\Delta = a^2+4b$ its discriminant. So, we can consider 3 cases :

If $\Delta > 0$, then we have $u_n = \alpha x_1^n + \beta x_2^n$, where α and β are two constants, and x_1 and x_2 are the solutions of (E). We assume that the absolute values of these two solutions are different. So we can take (eventually renaming these solutions) $|x_1| > |x_2|$. Thus, $u_n = x_1^n(\alpha + \beta\frac{x_2^n}{x_1^n}) \sim \alpha x_1^n$ and it follows that, for n great enough, $|\beta(\frac{x_2}{x_1})^n| < 1$ and thus $u_n = \lfloor \alpha x_1^n \rfloor + 1$ for n great enough. Thus, with an analogous proof that for question 4, $\mu(U)=0$.

If $\Delta = 0$ then we have $u_n = (\alpha+\beta n)x_0^n$. So $u_n \sim \beta n x_0^n$. Thus, for n great enough, $u_n > x_0^n$. So, using question 4 and the second lemma, $\mu(U)=0$.

If $\Delta < 0$ then we have $u_n = \rho^n \times C\cos(n\theta+\phi)$ with C, θ, ϕ and ρ reals and $\rho > 0$. We can take $\theta \in]0;2\pi[$ ($\theta = 0$ implies $\mu(U)=0$, as seen before). We can prove that $w_n = \cos(n\theta)$ diverges and so $v_n = C\cos(n\theta+\phi)$ diverges too. As we want $(u_n) > 0$ and we have $\rho > 0$, we must take $(v_n) > 0$. So, there exists $\epsilon > 0$ such that there is an infinity of $k\in\mathbb{N}$ such that $C\cos(k\theta+\phi) \geqslant \epsilon$. Thus

$$\limsup_{n\to+\infty} u_n \geqslant \limsup_{n\to+\infty} \epsilon\rho^n$$

Finally, the second lemma gives $\mu(U)=0$.

At any rate, $\mu(U)=0$.

Question 6

Let A the set of number which begin with "1" in trinary numeral system, and B the number which begin with "2" in this system. We obviously have $\mu_1(A \cup B) = \mu_1(\mathbb{N}) = 1$ and $\mu_1(A \cap B) = \mu_1(\varnothing) = 0$. We compute $\mu_1(A)$ and $\mu_1(B)$.

$$\mu_1(A) = \limsup_{n \to +\infty} \frac{\#(A \cap [1, n])}{n}$$

We take a subsequence of $u_n = \frac{\#(A \cap [1,n])}{n}$ using $n = 2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 3^k$. So, when n is big enough, k is big enough. We also have $\#(A \cap [1, n]) = 3^0 + 3^1 + ... + 3^{k-1} + 3^k$. Indeed, we count 3^i elements of A in $[3^i; 3^{i+1} - 1]$ for $i \in [0, k-1]$ (half of them begin with "1" ; the other half begin "2"), and 3^k elements of A in $[3^k, 2 \times 3^k]$ (because numbers that begin by "1" can be found before those that begin by "2"). Thus, we have :

$$\limsup_{n \to +\infty} u_n = \limsup_{k \to +\infty} \frac{\#(A \cap [1, 2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 3^k])}{2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 3^k}$$

$$\limsup_{n \to +\infty} u_n = \limsup_{k \to +\infty} \frac{3^0 + 3^1 + ... + 3^{k-1} + 3^k}{2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 3^k}$$

$$\limsup_{n \to +\infty} u_n = \limsup_{k \to +\infty} \frac{\frac{3^{k+1}-1}{2}}{2 \times \frac{3^k-1}{2} + 3^k}$$

$$\limsup_{n \to +\infty} u_n = \limsup_{k \to +\infty} \frac{1}{2} \frac{3^{k+1} - 1}{2 \times 3^k - 1}$$

$$\limsup_{n \to +\infty} u_n = \limsup_{k \to +\infty} \frac{1}{2} \times \frac{3}{2}$$

$$\limsup_{n \to +\infty} u_n = \frac{3}{4}$$

And then, since we take a subsequence, $\mu_1(A)$ is higher (or equal) than it (since μ_1 is defined by a superior limit). Thus : $\mu_1(A) \geqslant \frac{3}{4}$.

Now, for $\mu_1(B)$, we introduce $v_n = u_n = \frac{\#(B \cap [1,n])}{n}$ and we introduce its subsequence with $n = 2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 2 \times 3^k$. So, when n is bigh enough, k is big enough. We also have $\#(B \cap [1, n]) = 3^0 + 3^1 + ... + 3^{k-1} + 3^k$. Indeed, we count 3^i elements of B in $[3^i; 3^{i+1} - 1]$ for $i \in [0, k-1]$ (half of them begin with "1" ; the other half begin "2"), and 3^k elements of B in $[3^k, 3 \times 3^k]$ (remember that we goes until "$... + 2 \times 3^k$" and not just "$... + 3^k$"). Thus, we have :

$$\limsup_{n \to +\infty} v_n = \limsup_{k \to +\infty} \frac{\#(B \cap [1, 2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 3^k])}{2 \times 3^0 + 2 \times 3^1 + ... + 2 \times 3^{k-1} + 2 \times 3^k}$$

$$\limsup_{n\to+\infty} v_n = \limsup_{k\to+\infty} \frac{3^0+3^1+...+3^{k-1}+3^k}{2\times 3^0+2\times 3^1+...+2\times 3^{k-1}+2\times 3^k}$$

$$\limsup_{n\to+\infty} v_n = \frac{1}{2}$$

And, for the same reason as for u_n, we have $\mu_1(B) \geqslant \frac{1}{2}$. So, if A and B satisfy the formula, we have $1+0 \geqslant \frac{3}{4}+\frac{1}{2}$, which is absurd.

Question 7

We conjecture the following results :
The formula is wrong for μ_2, with the same counter-example as for μ_1.
The formula is wrong for μ_3, with the following sets : A the subset of numbers which begins by a "1" in trinary system if they have an even number of digits, and by a "2" in this system when they have an odd number of digits. And the same definition for B, but switching "1" and "2".

Question 8

We define two new densities:

$$\mu_4(E) = \limsup_{n\to+\infty} \frac{\sum_{k\in E\cap[1,n]} \frac{1}{k}}{\ln(n)}$$

and

$$\mu_{5,m}(E) = \limsup_{n\to+\infty} \frac{\sum_{k\in E\cap[1,n]} k^m}{\sum_{k\in[1,n]} k^m} \text{for any given } m \in \mathbb{N}\cup\{0\}.$$

We make few remarks :
First, for any natural subset E, we have

$$\mu_{5,0}(E) = \mu_1(E)$$

$$\mu_4(E) = \limsup_{n\to+\infty} \frac{\sum_{k\in E\cap[1,n]} \frac{1}{k}}{\sum_{k\in[1,n]} \frac{1}{k}}$$

since $\sum_{k=1}^{n} \frac{1}{k} \sim \ln(n)$.

Second, those two densities verifie the same properties that the μ_1, μ_2 and μ_3 in the questions 1,2,3 and 4. Indeed, for the question 1, the previous equivalent give $\mu_4(\mathbb{N}) = 1$ (we obviously have $\mu_{5,m}(\mathbb{N}) = 1$). The existence of superior limit achieve the proof, as for question 1.

For question 2, both densities could be expressed as a limit of a rational function, that has a finite numerator (as soon as $k > \max(E)$), and a denominator which can be as big as wanted. Then, we also have $\mu_4(E) = 0$ and $\mu_{5,m}(E) = 0$ when E is finite.

For question 3, we have :

$$\mu_4(E) = \limsup_{n\to+\infty} \frac{\sum_{k\in[1,\lfloor\frac{n-a}{d}\rfloor]} \frac{1}{a+kd}}{\sum_{k\in[1,n]} \frac{1}{k}} = \limsup_{n\to+\infty} \frac{1}{d} \frac{\sum_{k\in[1,\lfloor\frac{n-a}{d}\rfloor]} \frac{1}{\frac{a}{d}+k}}{\sum_{k\in[1,n]} \frac{1}{k}}$$

And, using the precedent equivalent :

$$\mu_4(E) = \limsup_{n\to+\infty} \frac{1}{d} \frac{\ln(\lfloor\frac{n-a}{d}\rfloor)}{ln(n)} = \limsup_{n\to+\infty} \frac{1}{d} \frac{\ln(\frac{n}{d})}{ln(n)} = \limsup_{n\to+\infty} \frac{1}{d}(1 - \frac{\ln(d)}{ln(n)}) = \frac{1}{d}.$$

And in the same way

$$\mu_{5,m}(E) = \limsup_{n\to+\infty} \frac{\sum_{k\cap[1,\lfloor\frac{n-a}{d}\rfloor]} (a+kd)^m}{\sum_{k\in[1,n]} k^m} = \limsup_{n\to+\infty} d^m \frac{\sum_{k\in\cap[1,\lfloor\frac{n-a}{d}\rfloor]} (\frac{a}{d}+k)^m}{\sum_{k\in[1,n]} k^m}$$

And we have, as a corollary of the Faulhaber formula : $\sum_{i=0}^{n} i^p \sim \frac{1}{p+1} n^{p+1}$, so :

$$\mu_{5,m}(E) = \limsup_{n\to+\infty} d^m \frac{\frac{1}{m+1}(\lfloor\frac{n-a}{d}\rfloor + 1)^{m+1}}{\frac{1}{m+1}(n+1)^{m+1}} = \limsup_{n\to+\infty} d^m (\frac{\frac{n}{d}}{n})^{m+1} = \frac{1}{d}.$$

For question 4, we notice that the formula of question 6 holds for μ_4 and $\mu_{5,m}$ when B is finite (some manipulations give the superior limit of a finite sum divides by an infinite one, as for μ). So, using the same argument that for μ, we have : $\mu_4(E) = \mu_{5,m}(E) = 0$, where E is a set of a geometric progression.

We also conjectured : $\forall (m,n) \in \mathbb{N}^2, \forall E \in \mathcal{P}(\mathbb{N}), \mu_{5,m}(E) = \mu_{5,n}(E)$.

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