Team France 3

1. Generalizing Perfectness

Abstract

The perfect numbers, which are equal to the sum of their strict divisors are studied since a long time by the mathematicians and there are still conjectures about them. In this problem, we extend this notion to "f-perfect" numbers, the classical perfect numbers being the "f-perfect" numbers for f the identity function. We use multiplicative functions and Dirichlet convolution in order to do that.

Concerning the problem, we treated almost completely the questions 1 to 4. For this we relied on the classical properties of some well known arithmetic functions.

We have also solved a part of the questions 5 and 6, and given a conjecture for this last. For the last questions, we were mostly looking for general properties of f-perfect or f-amicable numbers.

Definitions, notations :

In all this problem, excepted counter-indication, p and p' will be primes, if necessary different from 2 ; k and q will be reals (generally integers) ; all other variables will be natural integers. The notation "$a \mid\mid b$" will be used for "a divide absolutely b", that is to say : a and b are natural integers, $a \mid b$, and $a \neq b$.

We'll use the notion of multiplicative function, defined as follows :

$$\text{f is multiplicative} \Leftrightarrow \forall (a, b) \in \mathbb{N}^2 \text{ such that } gcd(a, b) = 1 \text{ we have}$$

$$f(ab) = f(a)f(b)$$

We'll also use :

$$\text{n is} f\text{-perfect} \Leftrightarrow f(n) = \sum_{d \mid\mid n} f(d)$$

or

$$\text{n is } f\text{-perfect} \Leftrightarrow 2f(n) = \sum_{d \mid n} f(d)$$

Question 1

a) τ-perfectness.

First we prove that the function τ is multiplicative. Indeed, let a and b be two coprime integers and $a = p_1^{\alpha_1} \times ... \times p_i^{\alpha_i}$ and $b = q_1^{\beta_1} \times ... \times q_j^{\beta_j}$ their canonical decomposition in prime factors ($p_1, \dots, p_i$ are distinct, as well as $q_1, \dots, q_j$, all the powers are strictly positive).

Then it is well-known (and easy to see) that $\tau(a) = (\alpha_1 + 1) \times ... \times (\alpha_i + 1)$ and $\tau(b) = (\beta_1 + 1) \times ... \times (\beta_j + 1)$. Since a and b are coprime, the decomposition of ab in prime factors is :

$$ab = p_1^{\alpha_1} \times ... \times p_i^{\alpha_i} \times q_1^{\beta_1} \times ... \times q_j^{\beta_j},$$

the p_k and q_l being all different.

Thus,

$$\tau(ab) = \tau(a) \times \tau(b).$$

Moreover, $\sum_{d|n} \tau(d)$ is a multiplicative function of n. Indeed, if a and b are two coprime natural numbers, we have :

$$\sum_{d|ab} \tau(d) = \sum_{d|a} \sum_{d'|b} \tau(dd')$$

Since τ is a multiplicative function :

$$\sum_{d|ab} \tau(d) = \sum_{d|a} \sum_{d'|b} \tau(d)\tau(d')$$

$$\sum_{d|ab} \tau(d) = (\sum_{d|a} \tau(d))(\sum_{d'|b} \tau(d'))$$

Let's now look for τ-perfect numbers.

First, 1 is clearly not τ-perfect.

Let $n = \prod_{i=1}^{m} p_i^{a_i}$ be the canonical decomposition of n in prime factors.

Analysis : Suppose n τ-perfect. Then :

$$2\tau(n) = \sum_{d|n} \tau(d)$$

$$2\tau(\prod_{i=1}^{m} p_i^{a_i}) = \sum_{d|\prod_{i=1}^{m} p_i^{a_i}} \tau(d)$$

$$2\prod_{i=1}^{m} \tau(p_i^{a_i}) = \prod_{i=1}^{m} \sum_{d|p_i^{a_i}} \tau(d).$$

As for any prime number p and any natural number α, $\tau(p^\alpha) = \alpha + 1$ this gives us

$$2\prod_{i=1}^{m}(a_i+1) = \prod_{i=1}^{m}\sum_{k=0}^{a_i}\tau(p_i^k)$$
$$2\prod_{i=1}^{m}(a_i+1) = \prod_{i=1}^{m}\sum_{k=0}^{a_i}k+1$$
$$2\prod_{i=1}^{m}(a_i+1) = \prod_{i=1}^{m}\frac{a_i(a_i+1)}{2}+(a_i+1)$$

And thus

$$2 = \prod_{i=1}^{m}\frac{a_i}{2}+1.$$

But, for any i such that $1 \leq i \leq m$, we have $\dfrac{a_i}{2}+1 \geq \dfrac{3}{2}$. Thus, if $m \geq 2$, then $\prod_{i=1}^{m}\frac{a_i}{2}+1 \geq (\frac{3}{2})^m \geq (\frac{3}{2})^2 > 2$, which is absurd. That means $m = 1$, and we have $\frac{a_1}{2}+1 = 2$ so $a_1 = 2$.

Then n is τ-perfect only if n is a square of a prime number.

Synthesis : Suppose $n = p^2$, p being prime. Then $\tau(p^2) = 3 = 1+2 = \tau(1)+\tau(p)$, and only 1 and p are strict divisors of n.

Finally, n is τ-perfect if and only if $n = p^2$, where p is a prime number.

b) $\tau + k$ perfectness

We start rephrasing the problem :

$$\text{n is } (\tau\text{-}1)\text{-perfect} \Leftrightarrow 2(\tau(n)-1) = \sum_{d|n}(\tau(d)-1)$$
$$\text{n is } (\tau\text{-}1)\text{-perfect} \Leftrightarrow 2\tau(n)-2 = \sum_{d|n}\tau(d)+\sum_{d|n}-1$$
$$\text{n is } (\tau\text{-}1)\text{-perfect} \Leftrightarrow 2\tau(n)-2 = \sum_{d|n}\tau(d)-\tau(n)$$
$$\text{n is } (\tau\text{-}1)\text{-perfect} \Leftrightarrow 3\tau(n)-2 = \sum_{d|n}\tau(d)$$

It is easy to see that 1 is (τ-1)-perfect.

Let $n = \prod_{i=1}^{m}p_i^{a_i}$ be the decomposition of $n \neq 1$ in prime factors.

Analysis : Suppose n $(\tau - 1)$-perfect. Like in a), and using again the fact that τ and $\sum_{d|n} \tau(d)$ are multiplicative, we get :

$$3\prod_{i=1}^{m}(a_i+1) - 2 = \prod_{i=1}^{m} \frac{a_i(a_i+1)}{2} + (a_i+1)$$
$$3 - \frac{2}{\prod_{i=1}^{m}(a_i+1)} = \prod_{i=1}^{m} \frac{a_i}{2} + 1$$

And $-\frac{2}{\prod_{i=1}^{m}(a_i+1)} < 0$ therefore $\prod_{i=1}^{m} \frac{a_i}{2} + 1 < 3$.

Again as in a), we find $m \leq 2$ and $a_1 < 4$.

Synthesis : By testing the different cases, we see that the only solution is $m = 1$ and $a_1 = 3$

Finally, n is $(\tau - 1)$-perfect if and only if $n = p^3$ or $n = 1$.

Let k be an arbitrary integer. Again, as before, we obtain :

$$n \text{ is } (\tau + k)\text{-perfect} \Leftrightarrow 2(\tau(n) + k) = \sum_{d|n}(\tau(d) + k),$$

and thus

$$n \text{ is } (\tau + k)\text{-perfect} \Leftrightarrow 2\tau(n) + 2k = \sum_{d|n} \tau(d) + k\tau(n).$$

It is easy to see that 1 is not $(\tau + k)$-perfect, except for $k = -1$.

First we assume $k \geq 1$.

Analysis : Suppose n $(\tau + k)$-perfect.

$$\text{n is } (\tau + k)\text{-perfect} \Leftrightarrow 2\tau(n) + (2 - \tau(n))k = \sum_{d|n} \tau(d)$$

And, as $\tau(n) \geq 2$, we obtain $(2-\tau(n))k \leq 0$, and so $\sum_{d|n} \tau(d) \leqslant 2\tau(n)$. As in the question a), after replacing again n by its decomposition in prime factors, the inequality becomes :

$$\prod_{i=1}^{m} \frac{a_i(a_i+1)}{2} + (a_i+1) \leq 2\prod_{i=1}^{m}(a_i+1)$$

and then

$$\prod_{i=1}^{m} \frac{a_i}{2} + 1 \leq 2.$$

This gives $m = 1$ and $a_1 \leq 2$, so $n = p$ or $n = p^2$, with p a prime.

Synthesis : If $n = p$:

$$\begin{aligned}
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow \tau(p) + k = \sum_{d||p} (\tau(d) + k) \\
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow 2 + k = \tau(1) + k \\
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow 2 = \tau(1)
\end{aligned}$$

So $n = p$ is not $(\tau + k)$-perfect.
If $n = p^2$:

$$\begin{aligned}
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow \tau(p^2) + k = \sum_{d||p^2} (\tau(d) + k) \\
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow 3 + k = \tau(1) + k + \tau(p) + k \\
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow 3 = 1 + 2 + k \\
\text{n is } (\tau + k)\text{-perfect} &\Leftrightarrow k = 0
\end{aligned}$$

This is absurd, hence $n = p^2$ is not $(\tau + k)$-perfect. Finally, there is no $(\tau + k)$-perfect integer for $k \geq 1$.

Now we study $k \leq -2$. We didn't find a formula giving all the $(\tau + k)$-perfect numbers in the genral case. But we made an algorithm which theoretically can find them for any k.

Analysis : Suppose n $(\tau + k)$-perfect.

$$\text{n est } (\tau + k)\text{-perfect} \Leftrightarrow (2 - k)\tau(n) + 2k = \sum_{d|n} \tau(d).$$

And as $2k < 0$ we have $\sum_{d|n} \tau(d) < (2 - k)\tau(n)$. In the same was as in question a, and using once again the decomposition of n in prime factors, we obtain :

$$\prod_{i=1}^{m} \left(\frac{a_i(a_i + 1)}{2} + (a_i + 1) \right) \leq (2 - k) \prod_{i=1}^{m} (a_i + 1)$$

and the division by $\prod_{i=1}^{m}(a_i+1) > 0$ on the both sides of this inequality yelds

$$\prod_{i=1}^{m} \frac{a_i}{2} + 1 \leq (2-k). \tag{1}$$

On the other hand, as $a_i \geq 1$ for any i, we have :

$$\left(\frac{3}{2}\right)^m \leq \prod_{i=1}^{m} \frac{a_i}{2} + 1 \leq (2-k).$$

And thus $m \leq \lfloor \frac{\ln(2-k)}{\ln(\frac{3}{2})} \rfloor$, because $m \in \mathbb{N}$.

On the other hand, in the inequality (1), $\frac{a_i}{2} + 1 > 1$ for any i, thus $\frac{a_i}{2} + 1 < 2 - k$, that yelds $a_i < 2 - 2k$.

The last two equations mean that we have only a finite number of cases to examine in order to find all the τ_k-perfect numbers. Thus we can make an algorithm that tests, for every possible values of m and a_i the following equality :

$$(2-k)\prod_{i=1}^{m}(a_i+1) + 2k = \prod_{i=1}^{m}\left(\frac{a_i(a_i+1)}{2} + (a_i+1)\right).$$

The simplest algorithm (and so not optimized at all) which treats this problem must do almost $(2-2k)^{\lfloor \frac{\ln(2-k)}{\ln(\frac{3}{2})} \rfloor}$ operations, which becomes too long, very fast, because the complexity is in order of $\mathcal{O}((ck)^{c' \ln(k)})$, where c and c' are constants. But theoretically we have a way to find all the $(\tau + k)$-perfect numbers for all k.

In pseudo-code, this algorithm would be of this form (it is really easy to improve it, for example noticing the commutativity of the equality to test) :

"For a_1 from 1 to $2-2k$
For a_2 from 1 to $2-2k$
...
For $a_{\lfloor \frac{\ln(2-k)}{\ln(\frac{3}{2})} \rfloor}$ from 1 to $2-2k$, Do :
If $(2-k)\prod_{i=1}^{m}(a_i+1) + 2k = \prod_{i=1}^{m} \frac{a_i(a_i+1)}{2} + (a_i+1)$
Then a_1, a_2, ... , $a_{\lfloor \frac{\ln(2-k)}{\ln(\frac{3}{2})} \rfloor}$ go in L
End If
End For
...

End For"

Here L is a list that stocks the a_i solutions, and that can be used to find all the n that are $(\tau + k)$-perfect.

Using this algorithm, we got the following results :
For $k = -2$, there is not $(\tau + k)$-perfect numbers.
For $k = -3$, n is $(\tau + k)$-perfect if and only if $n = p_1^2 p_2 p_3$.
For $k = -4$, there is not $(\tau + k)$-perfect numbers.
For $k = -5$, there is not $(\tau + k)$-perfect numbers.
For $k = -6$, n is $(\tau + k)$-perfect if and only if $n = p_1^3 p_2^2 p_3$.

Question 2

Let $n = 2^k a$, where $k \in \mathbb{N} \cup \{0\}$ and a is an odd integer.

It is well known that φ is multiplicative. We can also show, using cyclotomic polynomials or geometric sequences that $\sum_{d|n} \varphi(d) = n$, that is a classical property. (For example, one can find a proof on : http ://promenadesmaths.free.fr/fonction_phi.htm for these two proofs.)

Analysis : Suppose n φ-perfect. Then :

$$2\varphi(n) = \sum_{d|n} \varphi(d) = n$$

So n is even and so $k \geqslant 1$.

$$2\varphi(2^k a) = 2^k a$$

$$2\varphi(2^k)\varphi(a) = 2^k a$$

$$2(2^{k-1})\varphi(a) = 2^k a$$

$$\varphi(a) = a$$

Because $\varphi(2^k)$ is the number of numbers coprime with 2^k, between 1 and 2^k. This number are the odd numbers, and in $[|1, 2^k|]$ they are at the number of $\frac{2^k}{2}$.
The last equality implies $a = 1$. So $n = 2^k$, with $k \in \mathbb{N}$.

Synthesis : Let $n = 2^k$.

$$\text{n is } \varphi\text{-perfect} \Leftrightarrow 2\varphi(2^k) = \sum_{d|2^k} \varphi(d)$$

$$\text{n is } \varphi\text{-perfect} \Leftrightarrow 2(2^{k-1}) = 2^k$$

So $n = 2^k$ is φ-perfect.

Finally :

$$\text{n is } \varphi\text{-perfect} \Leftrightarrow n = 2^k$$

with $k \in \mathbb{N}$.

Question 3

a) $f(n) = n - 1$

Let $f(n) = n - 1$. Let $n = 2^k(2^{k+1} - 2k - 1)$, where $k \geq 1$, be such that $(2^{k+1} - 2k - 1) \in \mathcal{P}$, $\mathcal{P}$ being the set of prime integers. Let's compute $\sum_{d|n, d\neq n} f(d)$:

$$\sum_{d|2^k(2^{k+1}-2k-1)} d - 1 = \sum_{i=0}^{k} 2^i(2^{k+1} - 2k - 1) - 1 + \sum_{i=0}^{k} 2^i - 1$$

$$\sum_{d|2^k(2^{k+1}-2k-1)} d - 1 = (2^{k+1} - 2k - 1 + 1)(\sum_{i=0}^{k} 2^i) - 2(k+1)$$

$$\sum_{d|2^k(2^{k+1}-2k-1)} d - 1 = 2^{2k+2} - 2^{k+1} - 2^{k+2}k + 2k - 2k - 2$$

$$\sum_{d|2^k(2^{k+1}-2k-1)} d - 1 = 2(n - 1)$$

Thus, n is f-perfect.

b) $f(n) = n + q$

Let $f(n) = n + q$, where $q \in \mathbb{Q}$. Using the question a, we consider the numbers n of the form $2^k p$ where $k \geq 1$ and $p \in \mathcal{P} \setminus \{2\}$.

Analysis : Suppose n f-perfect. Then :

$$2(2^kp+q) = \sum_{d|2^kp}(d+q) = \sum_{i=0}^{k}(2^ip+q)+\sum_{i=0}^{k}(2^i+q) = (p+1)\sum_{i=0}^{k}2^i+2q(k+1),$$

that gives us

$2^{k+1}p = (p+1)(2^{k+1}-1) + 2qk = 2^{k+1}p = 2^{k+1}p - p + 2^{k+1} - 1 + 2qk$

and then

$$p = 2^{k+1} + 2qk - 1.$$

Synthesis : Suppose $n = 2^k(2^{k+1} + 2qk - 1)$ (and so $2qk \in \mathbb{Z}$). We have :

$$\begin{aligned}
\sum_{d|2^k(2^{k+1}+2qk-1)}(d+q) &= \sum_{i=0}^{k}2^i(2^{k+1}+2qk-1) + q + \sum_{i=0}^{k}2^i + q \\
&= (2^{k+1}+2qk-1+1)(\sum_{i=0}^{k}2^i) + 2q(k+1) \\
&= (2^{k+1}+2qk)(2^{k+1}-1) + 2qk \\
&= 2^{2k+2} + 2^{k+2}qk - 2^{k+1} \\
&= 2(n+q).
\end{aligned}$$

We found a sufficient condition :

$$n = 2^k(2^{k+1} + 2qk - 1) \Rightarrow \text{n is } f\text{-perfect.}$$

Question 4

Let $n = \prod_{i=1}^{m} p_i^{a_i}$ the decomposition of n in prime factors. Obviously 1 is ln-perfect.

Analysis : Suppose n ln-perfect. Then $\ln(n) = \sum_{d||n}\ln(d)$ and thus :

$$n = \prod_{d||n} d.$$

We suppose that n have at least one strict divisor d_0 different from 1 such that $d_0 \neq \frac{n}{d_0}$, that is to say $n \neq {d_0}^2$. Then

$$n = d_0\frac{n}{d_0}\prod_{d||n;d\neq d_0;d\neq\frac{n}{d_0}} d,$$

and thus

$$1 = \prod_{d||n; d\neq d_0; d\neq \frac{n}{d_0}} d.$$

So, as $d \mid\mid n \Rightarrow d \geq 1$, the only strict divisor of n different from d_0 and $\frac{n}{d_0}$ must be 1. So $d_0 = p$ et $\frac{n}{d_0} = p'$ prime numbers and so $n = pp'$, or $d_0 = p$ and $\frac{n}{d_0} = p^2$ and so $n = p^3$. Finally, $n = p$ or p^2 or pp' or p^3, where p and p' are prime.

Synthesis : If $n = p$, using the equality above, we get $p = 1$, which is absurd.
If $n = p^2$, then $p^2 = 1 \times p$, which is absurd.
If $n = p^3$, then $p^3 = 1 \times p \times p^2$, which is true.
If $n = pp'$, then $pp' = 1 \times p \times p'$, which is true.

Finally, n is ln-perfect if and only if $n = p^3$, or $n = pp'$ or $n = 1$ where p and p' are prime.

Question 5

Let $f(n) = (-1)^n$. Let $n = 2^k a$, with $k \in \mathbb{N}$ and a an odd number.

Analysis : Suppose n f-perfect. Let study first $k \geq 1$.

$$\begin{aligned}
2(-1)^{2^k a} &= \sum_{d|2^k a} (-1)^d \\
2 &= \sum_{i=0}^{k} \sum_{d|a} (-1)^{2^i d} \\
2 &= \sum_{d|a} (-1)^d + \sum_{i=1}^{k} \sum_{d|a} (-1)^{2^i d} \\
2 &= -\tau(a) + \sum_{i=1}^{k} \tau(a) \\
(k-1)\tau(a) &= 2
\end{aligned}$$

$\tau(a)$ and $k-1$ are positive integers, so we have :

$$k - 1 = 1; \tau(a) = 2 \Leftrightarrow k = 2; a = p$$

or

$$k-1=2; \tau(a)=1 \Leftrightarrow k=3; a=1$$

And so $n=4p$, with $p \in \mathcal{P}$.

Let study now the case when $k=0$.

$$2(-1)^a = \sum_{d|a}(-1)^d$$
$$-2 = -\tau(a)$$
$$\tau(a) = 2$$

Thus $a=p$ and so $n=p$, with $p \in \mathcal{P} \setminus \{2\}$.

Synthesis : If $n=8$, then $\sum_{d||8}(-1)^d = -1+1+1 = 1 = (-1)^8$. So $n=8$ is f-perfect.

Let $p \in \mathcal{P} \setminus \{2\}$. If $n=p$, then $\sum_{d||p}(-1)^d == (-1)^1 = (-1)^p$. So $n=p$ is f-perfect.

If $n=4p$,then
$\sum_{d||4p}(-1)^d = (-1)^1 + (-1)^2 + (-1)^4 + (-1)^p + (-1)^{2p} = (-1)^{4p}$.
So $n=4p$ is f-perfect.

Finally, n is f-perfect if and only if $n=p$ or $n=4p$ with p a prime.

Let $\omega \in \mathbb{C}$ a root of unity.
Let $f(n) = \omega^n$.
Let $n = m^k a$, with $k \in \mathbb{N}$, $m = \min\{i \in \mathbb{N}*/\omega^i = 1\}$, and a such as $gcd(m,a)$.

We treated only the case when $m=p$ and $k \geq 1$.

Analysis : Suppose n is f-perfect, we have :

$$2\omega^{m^k a} = \sum_{d|m^k a} \omega^{m^k d}$$

$$2 = \sum_{i=0}^{k} \sum_{d|a} \omega^{m^i d}$$

$$2 = \sum_{i=1}^{k} \sum_{d|a} \omega^{m^i d} + \sum_{d|a} \omega^{d}$$

$$2 = k\tau(a) + \sum_{d|a} \omega^{d}$$

If the equality holds, then the sum is real.
We have

$$\sum_{d|a} \omega^d \geq \sum_{d|a} (-1)$$

by the triangular inequality, and so $\sum_{d|a} \omega^d \geq -\tau(a)$.
Then $2 \geq -\tau(a) + k\tau(a)$, that is $2 \geq (k-1)\tau(a)$. Because $k-1$ and $\tau(a)$ are positive integers, we have :

$$k - 1 = 1; \tau(a) = 2 \Leftrightarrow k = 2; a = p$$

or

$$k - 1 = 2; \tau(a) = 1 \Leftrightarrow k = 3; a = 1$$

or

$$k - 1 = 1; \tau(a) = 1 \Leftrightarrow k = 2; a = 1$$

Finally $n = m^2 p$ or $n = m^3$ or $n = m^2$, with $p \in \mathcal{P} \setminus \{m\}$.

Synthesis : If $n = m^2$, we have $\omega^{m^2} = 1$
and $\sum_{d||m^2} \omega^d = \omega^1 + \omega^m = \omega + \omega^{m^2}$ Since $\omega \neq 0$, n is not f-perfect.

If $n = m^2 p$, we have :

$$\text{n is } f\text{-perfect} \Leftrightarrow \omega^{m^2 p} = \sum_{d||m^2 p} \omega^d$$

$$\text{n is } f\text{-perfect} \Leftrightarrow 1 = \omega^1 + \omega^m + \omega^{m^2} + \omega^p + \omega^{mp}$$

$$\text{n is } f\text{-perfect} \Leftrightarrow 1 = \omega + 1 + 1 + \omega^p + 1$$

$$\text{n is } f\text{-perfect} \Leftrightarrow -2 = \omega + \omega^p$$

Since $|\omega| = 1$, n is f-perfect if and only if $\omega = -1$.

If $n = m^3$, we have :

$$\begin{aligned}
&\text{n is } f\text{-perfect} \Leftrightarrow \omega^{m^3} = \sum_{d||m^3} \omega^d \\
&\text{n is } f\text{-perfect} \Leftrightarrow 1 = \omega^1 + \omega^m + \omega^{m^2} \\
&\text{n is } f\text{-perfect} \Leftrightarrow -1 = \omega
\end{aligned}$$

Again, n is f-perfect if and only if $\omega = -1$.

Thus, with the hypothesis $m = p$ and $k \geq 1$, there are f-perfect numbers if and only if $\omega = -1$.

Question 6

Let $m \in \mathbb{N}$, $m \geq 1$. Let $f(n) = \binom{m}{n}$ Remark that 1 isn't f-perfect for all m. And note that if $n > m$, $\binom{m}{n} = 0$ whereas 1 divides n and $\binom{m}{1} = m > 0$, hence n is not f-perfect. We first prove the following result :
if $m = p + 1$ with $p \in \mathcal{P}$, only $n = p$ is f-perfect.
In fact, consider n a f-perfect number, we have :

$$\binom{p+1}{n} = \sum_{d||n} \binom{p+1}{d}.$$

If k is an integer such that $2 \leq k \leq p - 1$, then

$$p \mid \binom{p+1}{k}.$$

In fact, we have : $(p+1)! = k! \times (p+1-k)! \times \binom{p+1}{k}$.
Since $k < p$ and $p + 1 - k < p$ and p is prime, p does not divide $(p + 1 - k)!$ nor k, so that, by Gauss's lemma, we have : $p \mid \binom{p+1}{k}$.
Therefore, if $2 \leq n \leq p - 1$, then

$$\sum_{d||n} \binom{p+1}{d} = 1 + \sum_{d||n, d>1} \binom{p+1}{d},$$

so p does not divide this sum, but $p \mid \binom{p+1}{n}$. Thus, the equality does not hold.

Else, if $n = p + 1$, then $\binom{p+1}{n} = 1$, whereas $2 \mid p + 1$ and $\binom{p+1}{2} > 1$. Since the terms of the sum are nonnegative, n can not be f-perfect. Else, if $n = p$, then

$$\binom{p+1}{p} = \binom{p+1}{1} = \sum_{d||n} \binom{p+1}{d}.$$

Thus, in this case, only p is n-perfect. Remark that only 2011 is f-perfect for $m = 2012$, which is a corollary of this result (since 2011 is prime).

We also conjectured the following result :
if $m \neq p+1$, then only 1 is an f-perfect integer then there is no perfect integer.

Question 7 : general and further results on f-perfectness

Algebraic structures and general results

We establish here some algebraic structures concerning the arithmetic functions, in order to prove some general results on f-perfectness, even if a part of the results here can be proven rather easily without using these structures.

To this, we can use the convolution, which seems to be particularly convenient for this problem. For two arithmetic functions f and g, it is defined by :

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$$

Hence :

$$\text{n is } f\text{-perfect} \Leftrightarrow 2f(n) = (f * 1)(n)$$

1 is here the function F such that for all natural n, $F(n) = 1$.

One can show that the set of arithmetic functions ($\mathcal{F}(\mathbb{N}, \mathbb{C})$ with the addition and convolution is a ring (proof in the annex). The neutral element is ϵ, whith $\epsilon(n) = 1$ if $n = 1$ and $\epsilon(n) = 0$ else. We will show easily some general results, often using the formula above.
First, let $n \in \mathbb{N}$ and E_n be the set of the arithmetic functions F such that n is F-perfect. let f and g two arithmetic functions and n a f-

perfect and g-perfect nonnegative integer. We have :

$$2f(n) = (f * 1)(n), 2g(n) = (g * 1)(n)$$
$$2f(n) + 2g(n) = (f * 1)(n) + (g * 1)(n)$$
$$2(f + g)(n) = ((f + g) * 1)(n)$$

Therefore, n is $(f + g)$-perfect.
And, considering k a real, we have :

$$2f(n) = (f * 1)(n)$$
$$2kf(n) = k(f * 1)(n)$$
$$2(kf)(n) = (kf * 1)(n)$$

So, n is (kf)-perfect. We can call this the "homogeneity property". Remark that one has always $(kf * g) = k(f * g)$, in fact, for all natural n :

$$(kf * g)(n) = \sum_{d|n}(kf)(d)g(\frac{n}{d})$$
$$(kf * g)(n) = \sum_{d|n} k \times f(d)g(\frac{n}{d})$$
$$(kf * g)(n) = k\sum_{d|n} f(d)g(\frac{n}{d})$$
$$(kf * g)(n) = k(f * g)(n)$$

Now, if n is f-perfect and $(f + k)$-perfect, with $k \neq 0$, we have :

$$\begin{cases} 2f(n) = (f * 1)(n) \\ 2(f(n) + k) = ((f + k) * 1)(n) \end{cases} \Rightarrow \begin{cases} 2f(n) = (f * 1)(n) \\ 2f(n) + 2k = (f * 1)(n) + k\tau(n) \end{cases}$$

$$\Rightarrow 2k = k\tau(n) \Rightarrow \tau(n) = 2$$

Therefore, if n is f-perfect and $(f + k)$-perfect then n is prime.

And, if n and all its divisors are f-perfect,let g an arithmetic function, we have :

$2f(n) = (f*1)(n) \Rightarrow (g*2f)(n) = (g*(f*1))(n) \Rightarrow 2(g*f)(n) = ((g*f)*1)(n)$,

thus n is $(g * f)$-perfect.

μ-perfectness

Let μ be the classical Mobius function, defined by $\mu(n) = 0$ if n is divisible by a perfect square (greater than 2), $\mu(n) = 1$ if n is the product of an even numbers of distinct primes, and $\mu(n) = -1$ else.
If n is μ-perfect, then $2\mu(n) = (\mu * 1)(n) = \epsilon(n)$. $n = 1$ is not μ-perfect. Else, if $n > 1$, then $\mu(n) = 0$ thus n is divisible by a perfect square. Reciprocal is obviously true. So, n is μ-perfect, if and only if n is divisible by a perfect square (different from 1).

ϵ-perfectness

If n is ϵ-perfect, then $2\epsilon(n) = (\epsilon * 1)(n) = 1(n) = 1$. Remark that 1 is not ϵ-perfect. If $n > 1$, then $\epsilon(n) = 0 \neq 1$ thus n is not ϵ-perfect either. So, there isn't any ϵ-perfect natural integer.

1-perfectness

$$\text{n is } f\text{-perfect} \Leftrightarrow 2 \times 1(n) = (1 * 1)(n) \Leftrightarrow 2 = \tau(n) \Leftrightarrow \text{n is prime}$$

"Perfect function"

We try to construct here a "perfect function", that is, a function f such that for all $n \geq 2$, n is f-perfect.
If $f(1) = 0$, one can easily show by a strong induction on n that $f(n) = 0$ for all natural n.
Else, let $f(1) = a$, where $a \neq 0$. By the homogeneity property, n is f-perfect if and only if n is $\frac{1}{a}f$-perfect, so that we can consider that $a = 1$. Remark that for all prime p,

$$f(p) = f(1) = 1.$$

And, a strong induction again on n shows that for any value of a, if a such function f exists, then there is only one possible value for $f(n)$.
Consider the case when n is a prime power, that is $n = p^m$ with p a prime and m a natural. We prove by a strong induction on m that $f(p^m) = 2^{m-1}$.
Basis at $m = 1$: we have $f(p) = a = 2^{1-1}$.
Inductive step : Suppose that for all k such that $1 \leq k \leq m$, $f(p^k) =$

2^{k-1}. We have :

$$f(p^{m+1}) = \sum_{d||p^{m+1}} f(d)$$
$$f(p^{m+1}) = \sum_{k=0}^{m} f(p^k)$$
$$f(p^{m+1}) = f(1) + \sum_{k=1}^{m} 2^{k-1}$$
$$f(p^{m+1}) = 1 + 2^m - 1$$
$$f(p^{m+1}) = 2^m$$

The conclusion follows.

We also make a conjecture for all the values of $f(n)$, thanks to some symbolic calculation. We think, for $n = \prod_{i=1}^{k} p_i^{a_i}$ (canonical decomposition in prime factors of $n > 1$) :

$$f(\prod_{i=1}^{k} p_i^{a_i}) = \sum_{m=1}^{\sum_{i=1}^{k} a_i} \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \prod_{i=1}^{k} \binom{a_i + j - 1}{a_i}$$

We think this function interesting because its values generalize the Bell numbers and the Stirling numbers of second kind.

Question 8

Definition and general properties

A "reasonnable" definition of the f-amicality could be this one, for n and m some positive integers :

$$\text{n and m are } f\text{-amicable} \Leftrightarrow \begin{cases} f(n) = \sum_{d||m} f(d) \\ f(m) = \sum_{d||n} f(d) \end{cases}$$

According to this definition, m and n are amicable if they are Id-amicable, Id being the identity function.

Likely to question 7, we obtain easily for m and n two natural integers :

$$\text{n and m are } f\text{-amicable} \Leftrightarrow f(n) + f(m) = (f * 1)(m) = (f * 1)(n)$$

Let m and n two f-amicable and g-amicable numbers, and $k \in \mathbb{R}$. We have :

$$\begin{aligned} & f(n) + f(m) = (f * 1)(m) = (f * 1)(n) \\ & kf(n) + kf(m) = k(f * 1)(m) = k(f * 1)(n) \\ & (kf)(n) + (kf)(m) = ((kf) * 1)(m) = ((kf) * 1)(n) \end{aligned}$$

Therefore, m and n are (kf)-amicable. And :

$$\begin{cases} f(n) + f(m) = (f * 1)(n) = (f * 1)(m) \\ g(n) + g(m) = (g * 1)(n) = (g * 1)(m) \end{cases}$$

$$f(n) + f(m) + g(n) + g(m) = (f * 1)(n) + (g * 1)(n) = (f * 1)(m) + (g * 1)(m)$$

$$(f + g)(n) + (f + g)(m) = ((f + g) * 1)(n) = ((f + g) * 1)(m)$$

Therefore, m and n are $f + g$-amicable.

Now we treat different classical arithmetic functions, m and n being always two natural integers.

ϵ-amicality

In this case, we have :

n and m are ϵ-amicable $\Leftrightarrow \epsilon(n) + \epsilon(m) = (\epsilon * 1)(m) = (\epsilon * 1)(n)$
n and m are ϵ-amicable $\Leftrightarrow \epsilon(n) + \epsilon(m) = 1$

Without loss of generality, m and n are ϵ-amicable if and only if $n = 1$ and $m \geq 2$.

φ-amicality

In this case, we have :

n and m are φ-amicable $\Leftrightarrow \varphi(n) + \varphi(m) = (\varphi * 1)(m) = (\varphi * 1)(n)$
n and m are φ-amicable $\Leftrightarrow \varphi(n) + \varphi(m) = m = n$

Thus $2\varphi(n) = n$, so that $2 \mid n$. Let $m = n = 2^a k$ where k is odd, and $a \in \mathbb{N}*$, we have :

$$\text{n and m are } \varphi\text{-amicable} \Leftrightarrow \varphi(2^a k) + \varphi(2^a k) = 2^a k$$
$$\text{n and m are } \varphi\text{-amicable} \Leftrightarrow 2\varphi(2^a)\varphi(k) = 2^a k$$
$$\text{n and m are } \varphi\text{-amicable} \Leftrightarrow 2(2^{a-1})\varphi(k) = 2^a k$$

Thus $\varphi(k) = k$, so that $k = 1$, and n and m are φ-amicable if and only if $m = n = 2^a$, with a a positive integer.

1-amicality

For $f = 1$:

$$\text{n and m are 1-amicable} \Leftrightarrow 1 + 1 = (1 * 1)(m) = (1 * 1)(n)$$
$$\text{n and m are 1-amicable} \Leftrightarrow 2 = \tau(m) = \tau(n)$$

Therefore, m and n are prime.

μ-amicality

For $f = \mu$:

$$\text{n and m are } \mu\text{-amicable} \Leftrightarrow \mu(n) + \mu(m) = (\mu * 1)(m) = (\mu * 1)(n)$$

$$\text{n and m are } \mu\text{-amicable} \Leftrightarrow \mu(n) + \mu(m) = \epsilon(m) = \epsilon(n)$$

If $m = 1$, then $n = 1$, but this doesn't work. If $m \neq 1$, $n \neq 1$ and :

$$\text{n and m are } \mu\text{-amicable} \Leftrightarrow \mu(n) + \mu(m) = 0$$

Thus n and m are μ-amicable if and only if n and m are divisible by the square of a prime number, or n is the product of an odd number of distinct primes, and m the product of an even number of distinct primes (or conversely, without loss of generality).

Additionnal remarks

It could be interesting to treat the problem of f-sociable numbers, that is to say, the integers n such that the sequence (u_k) defined by $u_0 = n$ et $f(u_{k+1}) = \sum_{d||u_k} f(d)$ is well-defined and periodic.

With this definition, and by some calculus very similar to the f-amicality, we have the following results :
$u_0, u_1, ..., u_n$ are ϵ-sociable if and only if n is odd and $u_0 > 1$, $u_1 = 1$, $u_2 > 1$, $u_3 = 1$,...,$u_n = 1$, without lack of generality.
If $u_0, u_1, ..., u_n$ are μ-sociable and none of them is equal to 1, they are all squares, or u_k is a product of a different primes, with $2 \mid k - a$, without lack of generality.
If $u_0, u_1, ..., u_n$ are 1-sociable, they are all primes.

Annex

Proof that $(\mathcal{F}(\mathbb{N}, \mathbb{C}), +, *)$ is a ring.
First,it is clear that $(\mathcal{F}(\mathbb{N}, \mathbb{C}), +)$ is a commutative group. Moreover, it is easy to see that $*$ is commutative and distributive towards $+$. Let's show that the function ϵ defined by $\epsilon(n) = 1$ if $n = 1$ and $\epsilon(n) = 0$ else., Let f be an arithmetic function. We have :

$$(\epsilon * f)(n) = \sum_{d|n} \epsilon(d) f(\frac{n}{d})$$
$$(\epsilon * f)(n) = \epsilon(1) f(\frac{n}{1})$$
$$(\epsilon * f)(n) = f(n)$$

Using the fact that $*$ is commutative, we can conclude that ϵ is the neutral element for $*$. To prove the associativity, let f,g and h be 3 arithmetic functions, we have :

$$(f * (g * h))(n) = \sum_{ab=n} f(a)(g * h)(b) = \sum_{ab=n} f(a) \sum_{cd=b} g(c)h(d)$$
$$(f * (g * h))(n) = \sum_{acd=n} f(a)g(c)h(d)$$

and :

$$((f * g) * h)(n) = \sum_{ab=n} (f * g)(a)h(b) = \sum_{ab=n} h(b) \sum_{cd=a} f(c)g(d)$$
$$((f * g) * h)(n) = \sum_{bcd=n} f(c)g(d)h(b)$$

Thus, $(f * (g * h))(n) = ((f * g) * h)(n)$.
Finally, $(\mathcal{F}(\mathbb{N}, \mathbb{C}), +, *)$ is a commutative ring.