Problem 8. Discharged Batteries

Team: France 2

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Abstract

In this paper, we have found how to find c charged batteries among n batteries, with 2-tries. We have decided to use graph theory to solve the first question with the Turán's theorem. We find that the minimum of tries to find 2 charged batteries among 5 charged and 5 uncharged with 2-tries is 7. The question 2 is unsolvable with Turán's theorem, since Turán's theorem deals essentially with graph not hypergraph. Finally, in question 3, we generalize the question 1, and remark that when $c \geq 3$ then the number of 2-tries equals c+2.

Question 1

We find a strategy in seven 2-tries. We place the batteries in four uniform groups. **(Figure 1)**

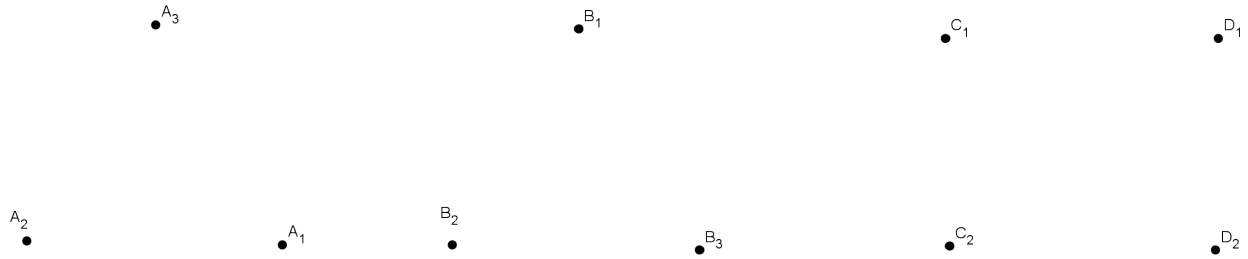

Figure 1: batteries are the points

First, we do all the possible tries with the triplet (A_1, A_2, A_3). It takes three tries. Obviously, there is at least two discharged batteries in this triplet. We do the same with the triplet (B_1, B_2, B_3). Now, we have done 6 tries. We do the 7th try with (C_1, C_2). There is exactly one discharged battery in this couple, because there is only five batteries discharged and four of them are in (A_1, A_2, A_3) and in (B_1, B_2, B_3). It remains only the couple (D_1, D_2). Since all the discharged batteries are in the other groups, in this couple there is only charged batteries.

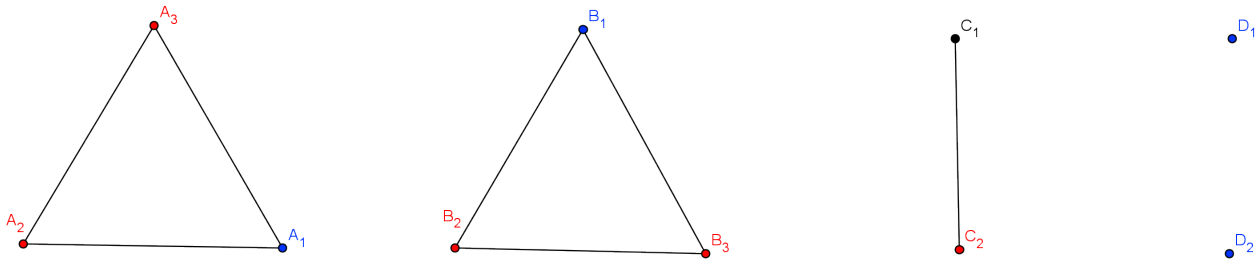

Figure 2: The blue batteries are the charged batteries, and the red are the discharged. The edges are the tries we did.

Let's prove that we can't find a strategy in less than seven 2-tries. Let's consider the simple graph where the vertices represent the batteries and the edges are the 2-tries we haven't made yet. At the beginning, it's a complete graph. **(figure 3)**. Each time we try a couple of batteries we erase the correspondant edge.

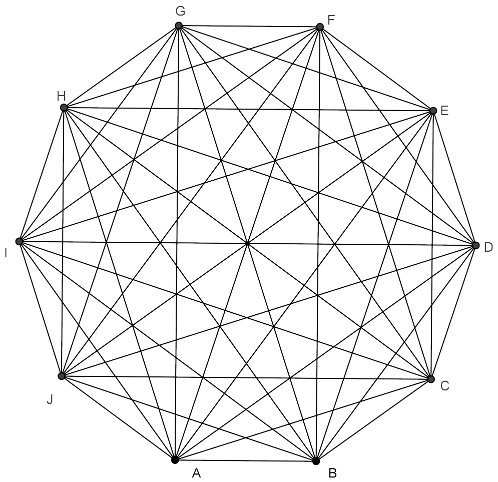

Figure 3: The graph at the beginning.

They are 5 charged batteries. The "good" tries are the edges of a precise pentagon in the graph. If after n 2-tries, there is no more pentagon, that is to say that we have found 2 charged batteries. The aim is to minimize the number of egdes erased for no longer having a pentagon, or in other word to maximize the number of edges without having a pentagon in the graph. Therefore, We refer to a problem of extremal graphs theory solved by Turán in 1941. We consider simple graphs G on the set of vertices $V = v_1, ..., v_n$, and a set of edges E. If v_i and v_j are vertices, we will note (v_i, v_j) the edge that has for origin v_i and for extremity v_j. A p-clique of G is a sub-graph complete of G with p edges, noted K_p.

Theorem.

If a graph $G = (V, E)$ with n edges doesn't have a p-clique, $p \geq 2$, then:

$$|E| \leq \binom{p-1}{2} \left(\frac{n}{p-1} \right)^2 .$$

In our case, p=5 and n=10. Thus:

$$|E| \leq \frac{75}{2}.$$

So, the maximum number of edges without having a pentagon is 37. We remember that the number of tries we have done is the number of edges erased starting with a complete graph with 10 vertices. We deduce that the number of tries we must do to find 2 charged batteries is $\binom{10}{2} - 37 = 8$. However, 8 is the number of 2-tries to power the car. To find 2 charged batteries, we don't need to do the last try.

Question 2

The Turán's theorem fo hypergraph hasn't existed yet. Therefore, we haven't used the idea we find in question 1. We have a strategy in 22 3-tries, nethertheless, we haven't proved is the best. You separate the batteries in 3 groups as shown in **figure 4**.

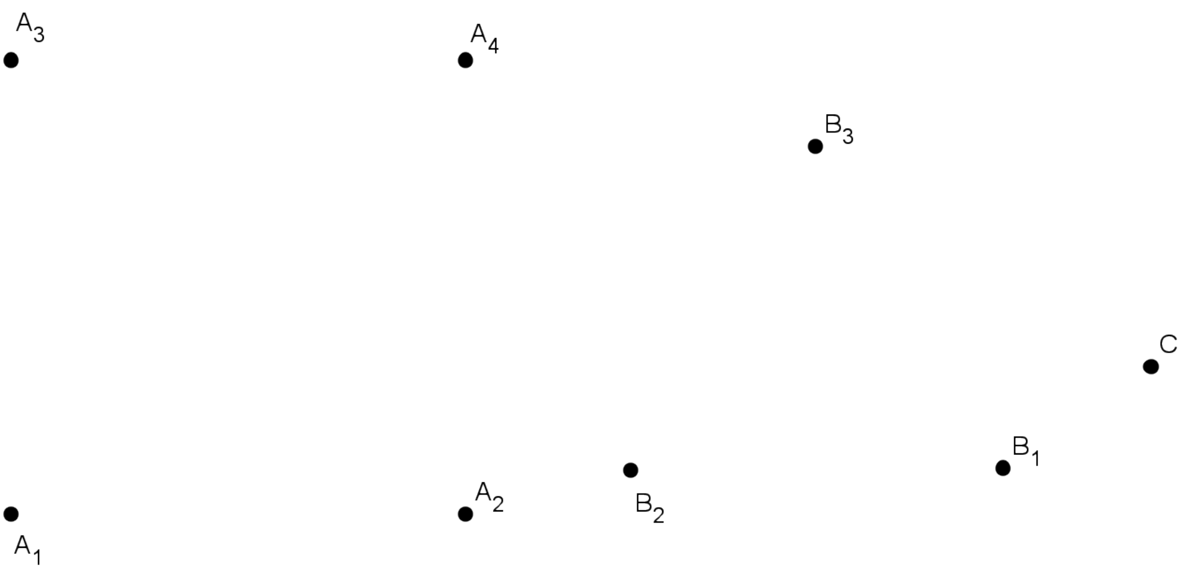

Figure 4

Then you try all the tries possible in the A-Group that is to say you make 4 3-tries. It's obvious that they are at least 2 discharged batteries. Then, you try (B_1, B_2, B_3), there is there at least 1 discharged battery. It's impossible that the four batteries in the A-group are discharged since there are only 4 discharged batteries and one is in the B-group. Thus, there is at least one charged battery in the A-group. With the same reasoning, we deduce that there is also at least one charged battery in the B-group. Suppose that C is charged. Then if we try it with 2 batteries of the A-group or 2 batteries of the B-group, we will find a triplet of charged batteries. We do these tries. It doesn't work. We deduce that C isn't charged. To sum up, there is 2 charged and 2 discharged in the A-group, 2 charged and 1 discharged in the B-group and C is discharged (**figure 5**).

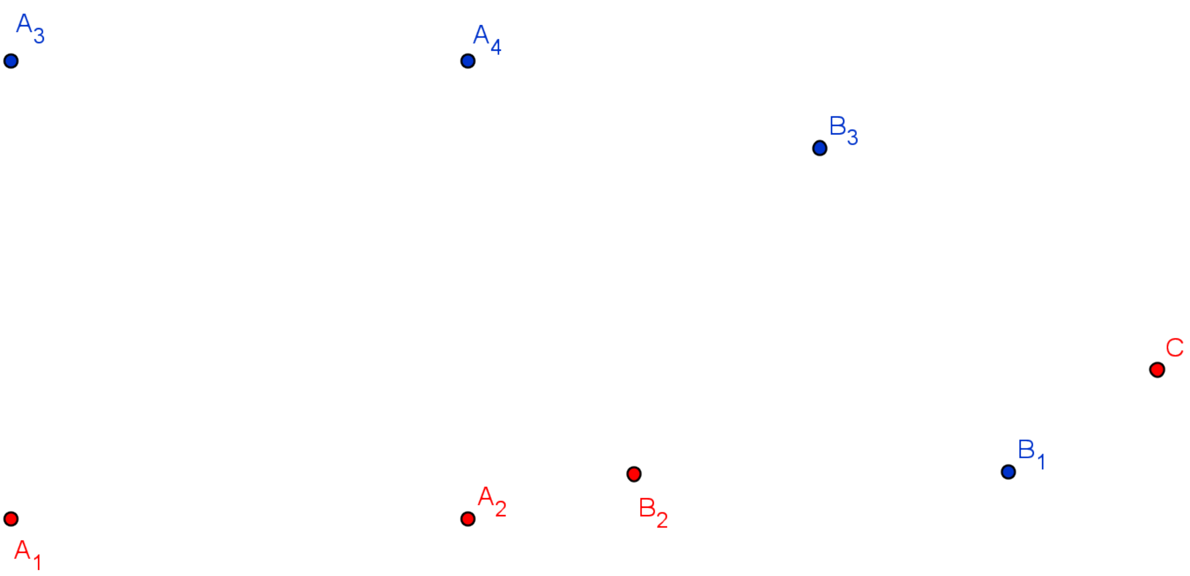

Figure 5: The blue batteries are the charged batteries, and the red are the discharged. The edges are the tries we did.

We try each couple of the B-group with 3 batteries of the A-group, due to the pigeonhole principle. In conclusion, with this strategy, the car run in 23 3-tries, and 22 3-tries is needed to find 3 charged batteries.

Question 3

Let N be the minimum number of 2-tries. To resume what we said in question 1,

$$N = \binom{c+u}{2} - \lfloor \binom{c-1}{2} \left(\frac{c+u}{c-1}\right)^2 \rfloor - 1$$

When c=u,

$$N = c(2c-1) - \lfloor \frac{(c-1)(c-2)}{2} \left(\frac{2c}{c-1}\right)^2 \rfloor - 1$$

$$N = c(2c-1) - \lfloor \frac{2(c-2)c^2}{c-1} \rfloor - 1$$

$$N = \lceil \frac{c(2c-1) - 2(c-2)c^2}{c-1} \rceil - 1$$

$$N = \lceil \frac{c(c+1)}{c-1} \rceil - 1$$

$$N = \lceil c + 2 + \frac{2}{c-1} \rceil - 1$$

For $c \geq 3$, $0 < \frac{2}{c-1} \leq 1$
Hence, if $c = u$ and $c \geq 3$, then $N = c + 2$.

The strategy is very simple. You have to refer to the complementary of the Turán's graph. You split the batteries in $c - 1$ groups uniforms and you try two batteries, only if they are in the same groups. The last possible try is the good one.

References

[1] ,Peter Keevash, Hypergraph Turán Problems, http://www.maths.qmul.ac.uk/ keevash/papers/turan-survey.pdf