Problem 6

France 2

Exercice 5: Recurrent sequences

Question 1

The following relations are direct consequences of the recurrence relation in the statement of the problem:

$$\forall n \geq 2, n u_{n+1} = \sum_{k=1}^{n} u_k^2 \tag{1}$$

$$n u_{n+1} = (n-1) u_n + u_n^2 \tag{2}$$

$$\forall n \geq 2, \quad \frac{u_{n+1}}{u_n} = \frac{1}{n}(n-1+u_n) \tag{3}$$

$$\forall n \geq 2, n(u_{n+1} - u_n) = u_n^2 - u_n = u_n(u_n - 1) \tag{4}$$

We will discuss the behavior of the sequence according to the position of u_3 with respect to 1.

• Consider first the case $0 < u_3 < 1$. We will prove by induction that $u_{n+1} < u_n < 1$ for $n \geq 3$. For $n = 3$, this follows from relation (3)

$$\frac{u_4}{u_3} = \frac{1}{3}(2 + u_3) < 1.$$

Assuming that $u_{n+1} < u_n < 1$, relation (4) shows that

$$(n+1)(u_{n+2} - u_{n+1}) < 0,$$

which implies the desired inequality $u_{n+2} < u_{n+1} < 1$.

Hence, when $0 < u_3 < 1$, the sequence $(u_n)_{n \in \mathbf{N}^*}$ is monotonic starting with rank 3 and bounded by 0 and $\max\{1, u_1, u_2\}$.Consequently $(u_n)_{n \in \mathbf{N}^*}$ is convergent.

• Consider next the case $u_3 = 1$. An immediate induction using relation (3) shows that $u_n = 1$ for $n \geq 3$, hence the sequence becomes constant from rank 3.

• Finally, consider the case $u_3 > 1$. We will prove by induction that $u_{n+1} > u_n > 1$ for $n \geq 3$. For $n = 3$ this follows from relation (3):

$$\frac{u_4}{u_3} = \frac{1}{3}(2 + u_3) > 1.$$

Assuming that $u_{n+1} > u_n > 1$, relation (4) yields $(n+1)(u_{n+2} - u_{n+1}) > 0$, hence $u_{n+2} > u_{n+1} > 1$.

We will prove that the sequence $(u_n)_{n\geq 3}$ is divergent. For $n \geq 3$, relation (4) shows that

$$u_{n+1} - u_n = \frac{u_n(u_n - 1)}{n} \geq \frac{u_3(u_3 - 1)}{n},$$

hence

$$u_{n+1} - u_3 \geq u_3(u_3 - 1) \sum_{k=3}^{n} \frac{1}{k}.$$

Since the last sequence is well-known to be divergent, it follows that $\lim_{n\to\infty} u_n = \infty$.

Question 2

In this case the recurrence relation becomes

$$\forall n \geq 3, \quad u_{n+1} = \frac{1}{n} \sum_{i=1}^{n} u_i u_{n+1-i}$$

0.1 $u_1^2 = u_2$

Then $u_n = u_1^n$. Let prove it by induction.

- Initialisation true since $u_1 = u_1$, $u_2 = u_1^2$, $u_3 = u_1 u_2 = u_1 u_1^2 = u_1^3$
- Induction: Assume for an integer $k \leq n-1$ where $n \geq 3$ that $u_k = u_1^k$. Moreover since $n - k \leq n - 1, u_{n-k} = u_1^{n-k}$. Therefore $u_k u_{n-k} = u_1^n$

$$\sum_{k=3}^{n-1} u_k u_{n-k} = (n-3)u_1^n$$

$$(n-1)u_n - u_1 u_{n-1} - u_2 u_{n-2} = (n-3)u_1^n$$

$$(n-1)u_n - u_1^n - u_1^n = (n-3)u_1^n$$

$$(n-3)u_n = (n-3)u_1^n$$

but $n - 3 \neq 0$ Then:

$$u_n = u_1^n$$

- Conclusion: $\forall n \geq 3, u_n = u_1^n$
- Then $(u_n)_{n\geq 1}$ is monotonic.
  - If $-1 < u_1 < 1$, $(u_n)_{n\geq 1}$ decreases and is bounded by u_1 and 0.
  - If $u_1 = 1$, $(u_n)_{n\geq 1}$ is constant and $\forall n \geq 1, u_n = 1$
  - If $u_1 > 1$, $(u_n)_{n\geq 1}$ grows, $u_n > u_1$ and $\lim_{n\to+\infty} u_n = +\infty$
  - If $u_1 < -1$, $(u_n)_{n\geq 1}$ is not monotonic, is divergent and has no limit.

0.2 $0 < u_2 < u_1^2$ and $0 < u_1$

Then $\forall n \geq 3, u_2^{\frac{n}{2}} < u_n < u_1^n$. Proof:

- Initialisation: $u_3 = u_1 u_2$ but $0 < u_2 < u_1^2$ Then: $u_2\sqrt{u_2} < u_3 < u_1^3$ i.e. $u_2^{\frac{3}{2}} < u_3 < u_1^3$.
- Induction: Assume for an integer k, $k \leq n-1$, $n \geq 3$, that $u_2^{\frac{k}{2}} < u_k < u_1^k$. Moreover since $n-k \leq n-1, u_2^{\frac{n-k}{2}} < u_{n-k} < u_1^{n-k}$. Then as all the terms of the two inegalities are positive:

$$u_2^{\frac{n}{2}} < u_k u_{n-k} < u_1^n$$

$$(n-1)u_2^{\frac{n}{2}} < \sum_{k=1}^{n-1} u_k u_{n-k} < (n-1)u_1^n$$

$$(n-1)u_2^{\frac{n}{2}} < (n-1)u_n < (n-1)u_1^n$$

  And as $n-1 > 0$,

$$u_2^{\frac{n}{2}} < u_n < u_1^n$$

- Conclusion:$\forall n \geq 3, u_2^{\frac{n}{2}} < u_n < u_1^n$

Then:

- If $u_2 > 1$, $(u_n)_{n\geq 1}$ is divergent, $\lim_{n\to+\infty} u_n = +\infty$ and $\forall n \geq 1, u_2^{\frac{1}{2}} < u_n$.
- If $u_1 < 1$, $(u_n)_{n\in N^*}$ is convergent, $\lim_{n\to+\infty} u_n = 0$ and $\forall n \geq 1, 0 < u_n \leq u_1$. More generally, we can prove by induction that $|u_n| \leq |u_1|^n$ which yields that when $-1 < u_1 \leq 0$, $(u_n)_{n\in N^*}$ is also convergent to 0.

0.3 $u_2 > u_1^2$

The same kind of proof shows that:

$$\forall n \geq 3, u_2^{\frac{n}{2}} > u_n > u_1^n$$

Then:

- If $u_1 > 1$, $(u_n)_{n\geq 1}$ is divergent, $\lim_{n\to+\infty} u_n = +\infty$ and $\forall n \geq 1, u_1 \leq u_n$.
- If $u_2 < 1$, $(u_n)_{n\in N^*}$ is convergent,$\lim_{n\to+\infty} u_n = 0$ and $\forall n \geq 1, 0 < u_n < u_2^{\frac{1}{2}}$. More generally, we prove by induction that $|u_n| \leq |u_2|^{\frac{n}{2}}$ what has the same consequences.

Generating functions

Consider the generating function $f(X) = \sum_{n\geq 1} u_n X^n$ of the sequence $(u_n)_n$. Using the recurrence relation, we compute formally

$$f(X)^2 = \sum_{n=2}^{+\infty} (\sum_{k=1}^{n-1} u_k u_{n-k}) X^n = u_1^2 X^2 + \sum_{n=3}^{+\infty} [(n-1)u_n] X^n = \sum_{n=3}^{+\infty} n u_n X^n$$

$$- \sum_{n=3}^{+\infty} u_n X^n + u_1^2 X^2 = X \sum_{n=3}^{+\infty} n u_n X^{n-1} - (f(X) - u_2 X^2 - u_1 X) + u_1^2 X^2.$$

Since $\sum_{n=3}^{+\infty} n u_n X^{n-1} = f'(X) - u_1 - 2u_2 X$, we obtain

$$\begin{aligned} f(X)^2 &= X(f'(X) - u_1 - 2u_2 X) - (f(X) - u_2 X^2 - u_1 X) + u_1^2 X^2 \\ &= X f'(X) - u_2 X^2 + u_1 X^2 - f(X) = X f'(X) - f(X) + a X^2, \end{aligned}$$

where $a = u_1^2 - u_2$.

Next, we rewrite the last relation as

$$\frac{f(X) - X f'(X)}{f(X)^2} = a \frac{X^2}{f(X)^2} - 1 \Leftrightarrow \left(\frac{X}{f(X)} \right)' = a \left(\frac{X}{f(X)} \right)^2 - 1.$$

We will assume that $u_1 \neq 0$. Then $\frac{X}{f(X)} = g(X)$ makes sense as formal series and the previous relation can be written

$$g'(X) = a g(X)^2 - 1.$$

0.3.1 $a = 0$ i.e. $u_2 = u_1^2$

Then:

$$g'(X) = -1 \Longleftrightarrow g(X) = -X + c, \quad c = \frac{1}{u_1}$$

$$\Longleftrightarrow f(X) = \frac{X}{c - X}$$

$$\Longleftrightarrow f(X) = -1 + \frac{1}{1 - \frac{X}{c}}$$

$$\Longleftrightarrow f(X) = -1 + \sum_{n=0}^{+\infty} \left(\frac{X}{c}\right)^n$$

$$f(X) = \sum_{n=1}^{+\infty} (\frac{1}{c})^n X^n = \sum_{n=1}^{+\infty} u_1^n X^n$$

Then: $\forall n \geq 1, u_n = u_1^n$ We find again the result of the first part 2.1 .

0.3.2 $a < 0$ i.e. $u_2 > u_1^2$

$$g'(X) = ag(X)^2 - 1 \Rightarrow (1) or (2)$$

(1) $(1) \Longrightarrow g(X) = \frac{-1}{\sqrt{-a}} tan(\sqrt{-a}X + c), \quad c = tan^{-1}(-\frac{\sqrt{-a}}{u_1})$

(2) $(2) \Longrightarrow g(X) = \frac{1}{\sqrt{-a}} \frac{1}{tan(\sqrt{-a}X+c)}, \quad c = tan^{-1}(\frac{u_1}{\sqrt{-a}})$

0.3.3 $a > 0$ i.e. $u_2 < u_1^2$

$$g'(X) = ag(X)^2 - 1 \Rightarrow (3) or (4)$$

(3) $(3) \Longrightarrow g(X) = \frac{-1}{\sqrt{a}} th(\sqrt{-a}X + c), \quad c = th^{-1}(-\frac{\sqrt{a}}{u_1})$

(4) $(4) \Longrightarrow g(X) = \frac{-1}{\sqrt{a}} \frac{1}{th(\sqrt{-a}X+c)}, \quad c = th^{-1}(-\frac{u_1}{\sqrt{a}})$

Question 4

a)

$$
\begin{aligned}
(n+1)u_n &= \sum_{i=1}^{n+1} u_i u_{n+2-i} \\
&= 2u_1 u_{n+1} + \sum_{i=2}^{n} u_i u_{n+2-i} \\
2u_1 u_{n+1} &= (n+1)u_n - \sum_{i=2}^{n} u_i u_{n+2-i} \qquad (1)
\end{aligned}
$$

Generating functions Considering the generating function $f(X) = \sum_{n\geq 1} u_n X^n$ we obtain:

$$
\begin{aligned}
f(X)^2 &= \sum_{n\geq 2}(\sum_{k=1}^{n-1} u_k u_{n-k})X^n \\
&= \sum_{n\geq 6}(\sum_{k=1}^{n-1} u_k u_{n-k})X^n + \sum_{n=2}^{5}(\sum_{k=1}^{n-1} u_k u_{n-k})X^n \\
&= \sum_{n\geq 4}(\sum_{k=1}^{n+1} u_k u_{n+2-k})X^{n+2} + X^2 \sum_{n=2}^{5}(\sum_{k=1}^{n-1} u_k u_{n-k})X^{n-2} \\
&= X^2 \sum_{n\geq 4}(\sum_{k=1}^{n+1} u_k u_{n+2-k})X^{n} + X^2 \sum_{n=2}^{5}(\sum_{k=1}^{n-1} u_k u_{n-k})X^{n-2} \\
\left[\frac{f(X)}{X}\right]^2 &= \sum_{n\geq 4}(n+1)u_n X^n + \sum_{n=2}^{5}(\sum_{k=1}^{n-1} u_k u_{n-k})X^{n-2}
\end{aligned}
$$

And as: $f'(X) = \sum_{n\geq 1} n u_n X^{n-1}$, $f(X)+Xf'(X) = \sum_{n\geq 1}(n+1)u_n X^n$.Then:

$$
\begin{aligned}
\left[\frac{f(X)}{X}\right]^2 &= f(X) + Xf'(X) - \sum_{n=1}^{3}(n+1)u_n X^n + \sum_{n=2}^{5}(\sum_{k=1}^{n-1} u_k u_{n-k})X^{n-2} \\
&= f(X) + Xf'(X) + a + bX + cX^2 + dX^3
\end{aligned}
$$

Where: $a = u_1^2; b = 2u_1u_2 - 2u_1; c = 2u_1u_3 + u_2^2 - 3u_2; d = 2(u_1u_4 + u_2u_3) - 4u_3$. Finally,

$$
\left[\frac{f(X)}{X}\right]^2 = f(X) + Xf'(X) + P(X), \quad P(X) = a + bX + cX^2 + dX^3
$$

Consider g(X)= Xf(X). Then:

$$\frac{g(X)^2}{X^4} = g'(X) + P(X)$$

Then,

$$\frac{g'(X)}{g(X)^2} = \frac{1}{X^4} - \frac{P(X)}{g(X)^2}$$
$$\frac{1}{g(X)} = \frac{1}{3X^3} + \int \frac{P(X)}{g(X)^2} dX + \alpha, \quad \alpha \in \mathbb{R}$$

If $u_1 = u_2 = u_3 = u_4 = 0$ an immediate induction shows that $\forall n \geq 4, u_n = 0$.

b)

Assume that $u_1 = u_2 = u_3 = 1$ and $0 \leq u_4 < 1$. Then, using (1):

$$2u_{n+1} = (n+1)u_n - 2(u_n + u_{n-1}) - \sum_{i=4}^{n-2} u_i u_{n+2-i}$$