Problem 4

France 2

We completely solved question 1), 2), 3), 4), 5).
We partially solved questions 6) and 7) by studying the cases $n = 1$, $n = 2$.

0.1 Question 1

Let ABC be a triangle and let CI and BJ be two internal angle bisectors. We will prove that they have the same length if and only if $AB = AC$. To shorten notations, we will denote

$$BC = a, \quad CA = b, \quad AB = c.$$

Using the angle bisector theorem, we have

$$\frac{AJ}{c} = \frac{JC}{a} = \frac{AJ + JC}{a + c} = \frac{b}{a + c},$$

hence $AJ = \frac{bc}{a+c}$ and $JC = \frac{ab}{a+c}$.

Using Stewart's theorem in triangle ABC with cevian BJ, we obtain

$$b(BJ^2 + AJ \cdot JC) = a^2 AJ + c^2 CJ,$$

which combined with the previous formulae for AJ and JC yields

$$BJ^2 + \frac{b^2 ac}{(a + c)^2} = ac.$$

By symmetry, we also obtain the relation

$$CI^2 + \frac{c^2 ab}{(a + b)^2} = ab.$$

These two relations make clear that $CI = BJ$ if $b = c$.

We will next prove that if $b < c$, then $CI < BJ$, which, by symmetry, will be enough to prove the converse. So, assume that $b < c$. Then $\frac{1}{(a+c)^2} < \frac{1}{(a+b)^2}$ and so, since $a + b > c$, we obtain

$$1 - \frac{b^2}{(a + c)^2} > 1 - \frac{c^2}{(a + b)^2} > 0.$$

Therefore

$$BJ^2 = ac \left(1 - \frac{b^2}{(a + c)^2}\right) > ab \left(1 - \frac{c^2}{(a + b)^2}\right) = CI^2,$$

which establishes the desired inequality.

0.2 Question 2

We will use again the notations

$$BC = a, \quad CA = b, \quad AB = c.$$

First, we prove the following special case of Steiner's theorem:

Lemma 0.1. *Let ABC be a triangle and let BH, respectively BJ be the median, respectively symmedian at B. Then*

$$\frac{AJ}{c^2} = \frac{CJ}{a^2} = \frac{b}{a^2 + c^2}.$$

Proof. Let x be the angle ABH and let y be the angle HBJ. By definition of the symmedian, the angle CBJ equals x. Next, the sine law in triangles ABH and CBH yields

$$\frac{AH}{\sin x} = \frac{BH}{\sin A}, \quad \frac{CH}{\sin(x+y)} = \frac{BH}{\sin C}.$$

Taking into account that $AH = CH$ and using the sine law in ABC, the previous two equalities yield

$$\frac{\sin(x+y)}{\sin x} = \frac{c}{a}.$$

Next, the sine law in triangles ABJ and CBJ gives

$$\frac{AJ}{\sin(x+y)} = \frac{BJ}{\sin A}, \quad \frac{JC}{\sin x} = \frac{BJ}{\sin C} \Rightarrow$$

$$\frac{AJ}{JC} = \frac{\sin(x+y)}{\sin x} \cdot \frac{c}{a} = \frac{c^2}{a^2},$$

which is equivalent to the desired equality. □

Using Stewart's theorem in triangle ABC with cevian BJ and the previous lemma, we obtain:

$$b(BJ^2 + AJ \cdot JC) = a^2 \cdot AJ + c^2 \cdot CJ \Rightarrow BJ^2 = \frac{2a^2c^2}{a^2+c^2} - \frac{a^2b^2c^2}{(a^2+c^2)^2}.$$

By symmetry, we obtain the analogous formula for the symmedian CI at C, namely:

$$CI^2 = \frac{2a^2b^2}{a^2+b^2} - \frac{a^2b^2c^2}{(a^2+b^2)^2}.$$

It is obvious from these two formulae that $CI = BJ$ if $b = c$. We will prove the converse. It is enough to prove that $BJ > CI$ when $c > b$. Or, if $c > b$, then an immediate computation shows that

$$\frac{2a^2c^2}{a^2+c^2} > \frac{2a^2b^2}{a^2+b^2}$$

and

$$-\frac{a^2b^2c^2}{(a^2+c^2)^2} > -\frac{a^2b^2c^2}{(a^2+b^2)^2}.$$

Adding these two inequalities and using the previous expressions of CI^2 and BJ^2, we obtain $BJ^2 > CI^2$, which is what we needed.

0.3 Question 3

Note first that it is not really clear what is meant by the length of an external bisector of an isosceles triangle. Indeed, if $AB = AC$, then the external bisector at A is parallel to the side BC, so it never meets it. However, when the triangle is not isosceles, it makes sense to consider the length of its external bisectors and to ask whether two such lengths can be equal. This is the problem that we consider and we will construct a non isosceles triangle having two equal external bisectors.

Consider the triangle ABC for which $\widehat{ABC} = 132$, $\widehat{ACB} = 12$ and $\widehat{BAC} = 36$ (all measures being in degrees). Let, as before BD and CE be the external bisectors at B and C. We claim that $BD = BC = CE$, which shows that $BD = CE$ and concludes the proof.

To prove that $BD = BC$, note that $\widehat{ABD} = \frac{180-132}{2} = 24$ and so

$$\widehat{CDB} = 180 - (12 + 132 + 24) = 12 = \widehat{BCD},$$

which yields the desired result.

To prove that $BC = CE$, note that $\widehat{CBE} = 180 - 132 = 48$ and $\widehat{BCE} = \frac{180-12}{2} = 84$, so that $\widehat{BEC} = 180 - (48 + 84) = 48 = \widehat{CBE}$, yielding again the desired result.

0.4 Question 4

With the same caveat as in the previous question, we will consider only non isosceles triangles and ask whether two ex-symmedians can be equal.

We will compute the length of the ex-symmedian at A assuming that ABC is not isosceles at A, so that $b \neq c$ (recall that $b = AC$ and $c = AB$). Let AA' be the external bisector at A and let AA'' be an ex-symmedian. Assume without loss of generality that $B \geq C$, so that B is between A'' and C. Note that

$$\widehat{CA'A} = \pi - (C + \frac{\pi + A}{2}) = B - \frac{\pi - A}{2},$$

so by definition of ex-symmedians we get

$$\widehat{A''AB} = \frac{\pi - A}{2} - x = C.$$

Next, use the sine law in triangles $A''AC$ and $A''AB$ to obtain

$$\frac{A''C}{\sin(A + C)} = \frac{AA''}{\sin C}, \quad \frac{A"B}{\sin C} = \frac{A''A}{\sin(\pi - B)}.$$

Combining this with the sine law in ABC yields

$$\frac{A''C}{A''B} = \frac{\sin^2 B}{\sin^2 C} = \frac{b^2}{c^2}.$$

One can easily check that this relation also holds if $B \leq C$ (the same computations apply, but this time C is between B and A'').

Lemma 0.2. *The length of the ex-symmedian at A is $\frac{abc}{|b^2-c^2|}$.*

Proof. Assume without loss of generality that $b > c$. We saw that $\frac{A''C}{A''B} = \frac{b^2}{c^2}$. Since $A''C = A''B + a$, we obtain $A''C = \frac{ab^2}{b^2-c^2}$ and $A''B = \frac{ac^2}{b^2-c^2}$. On the other hand, Stewart's theorem in triangle $AA''C$ with cevian AB yields

$$c^2 A''C + aA''C \cdot A''B = a \cdot (AA'')^2 + A''B \cdot b^2.$$

As $c^2 \cdot A''C = b^2 \cdot A''B$, we obtain

$$AA'' = \sqrt{A''C \cdot A''B} = \frac{abc}{b^2 - c^2},$$

finishing the proof. □

It is now clear that we can construct many non isosceles triangles having two equal ex-symmedians. Indeed, it is enough to ensure that $a^2 - b^2 = b^2 - c^2$, for instance.

0.5 Question 5

The fact that the internal bisectors are 1-interior segments is well-known, by the angle bisector theorem (which is just an application of the sine law). The fact that symmedians are 2-interior segments was established while solving question 2 (see lemma 0.1 and note that it also yields $AJ < AC = b$, that is J is between A and C).

Let AA' be the external bisector at A. It is clear that A' is not on the segment $[BC]$. Without loss of generality, assume that B is between A' and C. Then the sine rule in triangles $AA'B$ and $AA'C$ yields

$$\frac{BA'}{\sin \frac{\pi - A}{2}} = \frac{AA'}{\sin B}, \quad \frac{CA'}{\sin \frac{\pi + A}{2}} = \frac{AA'}{\sin C}.$$

Comparing the two relations and using that $\frac{\sin B}{\sin C} = \frac{b}{c}$ and $\sin \frac{\pi - A}{2} = \sin \frac{\pi + A}{2}$ yields $\frac{BA'}{CA'} = \frac{c}{b}$, proving that external bisectors are 1-exterior segments.

The fact that the ex-symmedians are exterior 2-segments was proved in question 4.

0.6 Question 6

Let AA_n be an internal n-segment. Using Stewart's theorem in triangle ABC with cevian AA_n we obtain

$$a(AA_n^2 + BA_n \cdot CA_n) = b^2 \cdot BA_n + c^2 \cdot CA_n,$$

so that using the definition of interior n-segment we obtain

$$AA_n^2 = \frac{b^2 c^n + c^2 b^n}{b^n + c^n} - \frac{a^2 b^n c^n}{(b^n + c^n)^2}.$$

This already makes it clear that isosceles triangles have equal n-interior segments. We saw that the converse holds for $n = 1$ or $n = 2$, but we do not believe that it holds in general.

0.7 Question 7

As we explained in questions 3 and 4, one has to be a little bit careful when asking the question, since the problem as stated does not make too much sense. In any case, we exhibited non isosceles triangles with equal 1-exterior and 2-exterior segments in questions 3 and 4.

Isosceles triangles's Figures

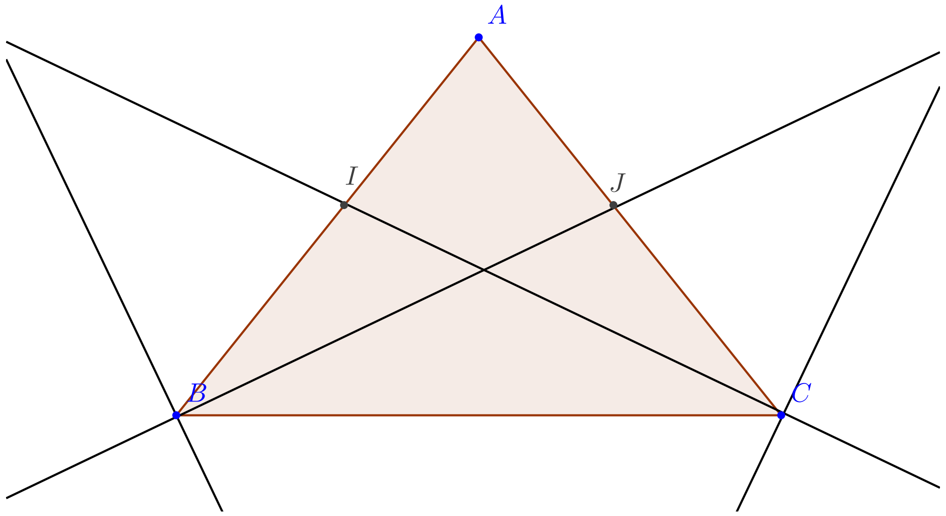

Figure 1: in the triangle ABC, let (BJ) be the bissector at B, let (CI) be the bissector at C. We set: $x = \widehat{JBC}$ and $y = \widehat{ICB}$

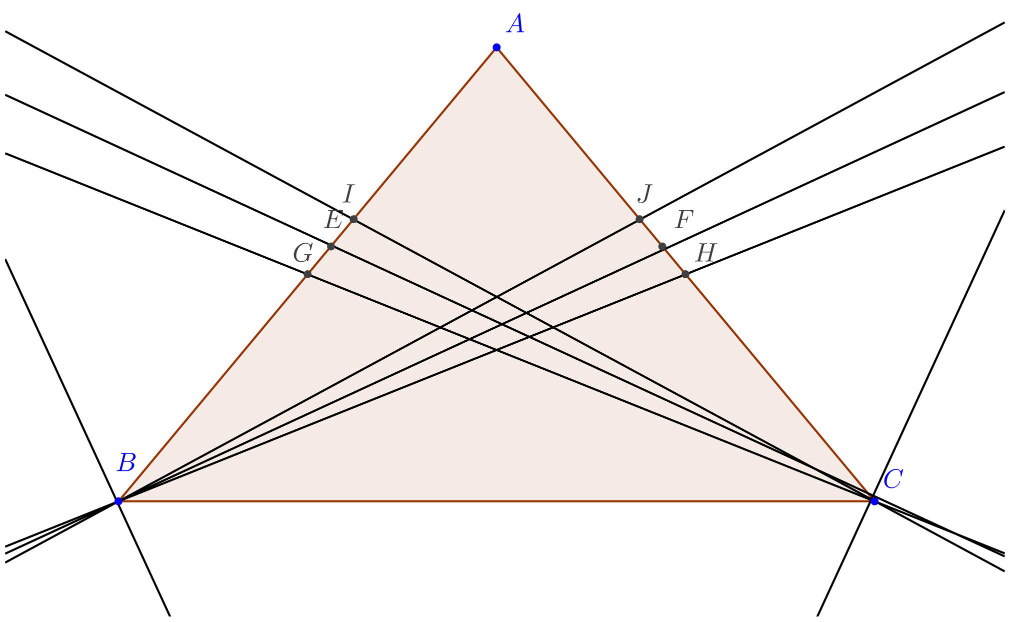

Figure 2: in the triangle ABC, let (BH), (BF) and (BJ) the medians, bissectors and symmedians at B, let (CG), (CE) and (CI) the medians, bissectors and symmedians at C.

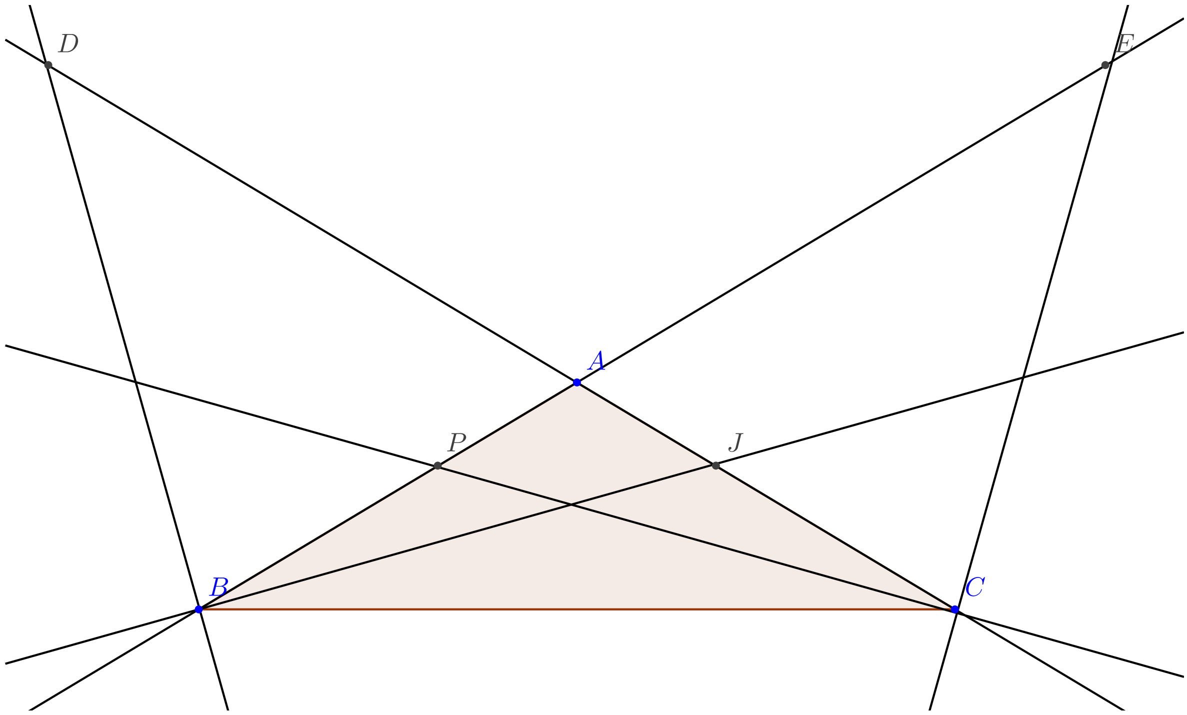

Figure 3: in the triangle ABC, let (BJ) and (BD) the internal and external bissectors at B, let (CP) and (CE) the internal and external bissectors at C. The triangles BJD and CPE are both rectangle respectively at B and at C.

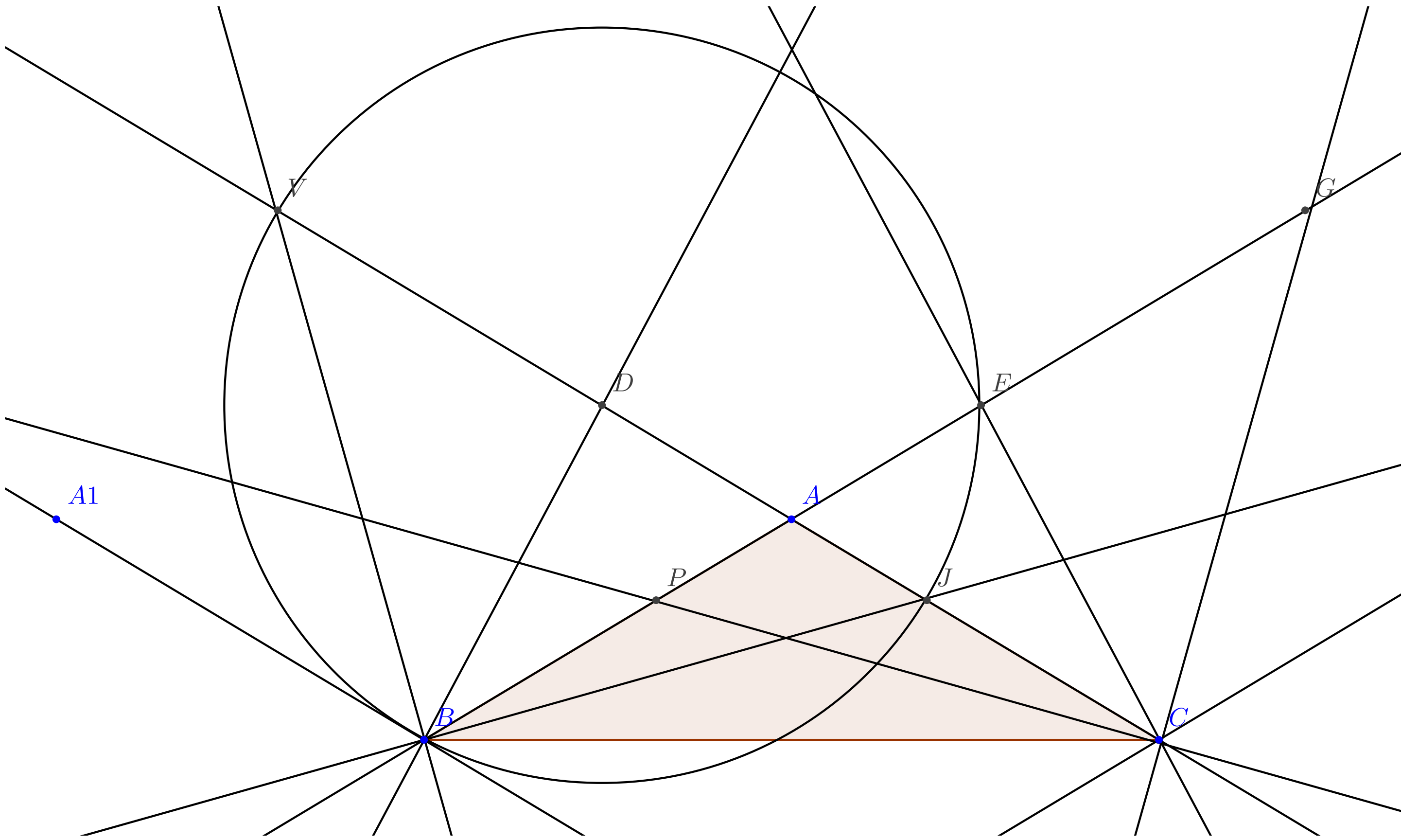

Figure 4: in the triangle ABC, let (BJ) and (BV) the internal and external bissectors at B, let (CP) and (CG) the internal and external bissectors at C. The triangles BJV and CPG are both rectangle respectively at B and at C.