PROBLEM 10

FRANCE 2

We solved completely questions 1), 2), 3), 4) and 6). We solved question 7) for μ_2.

We say that a set E has a density for μ_i if the limsup is in fact a limit. We prove that if a set has a density for μ_1 then it has a density for μ_2 and μ_3, which enables us to solve imediately questions 1), 2), 3), 4), because in that case it suffices to study μ_1 by our lemma, which is easy.

In question 6) we gave a counterexample which relies on the fact that we found a subset which has no density for μ_1. In fact this subset doesn't have density for μ_2 (which is stronger by the lemma).

For any $E \subseteq \mathbb{N}$, define three real numbers :

$$\mu_1(E) = \limsup_{n\to\infty} \frac{\sharp(E \cap \mathbb{N})}{n}$$
$$\mu_2(E) = \limsup_{x\to 1^-} (1-x) \sum_{n\in E} x^n$$
$$\mu_3(E) = \limsup_{x\to 1^+} (x-1) \sum_{n\in E} \frac{1}{n^x}$$

We will say that E has a density for μ_i if the limsup is in fact a limit.

QUESTION 1

Let's show that $\forall E \subseteq \mathbb{N}$:

$$0 \leq \mu_1(E) \leq 1$$
$$0 \leq \mu_2(E) \leq 1$$
$$0 \leq \mu_3(E) \leq 1$$

It's obvious that :

$$\forall E \subseteq \mathbb{N}, \mu_1(E) \geq 0, \mu_2(E), \geq 0, \mu_3(E) \geq 0$$

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$$\forall E \subseteq \mathbb{N}, \forall n \in \mathbb{N} \frac{\sharp(E \cap \mathbb{N})}{n} \leq \frac{n}{n} \leq 1$$

So

$$\limsup_{n\to\infty} \frac{\sharp(E \cap \mathbb{N})}{n} \leq 1$$

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$$\forall E \subseteq \mathbb{N}, \forall x \geq 0, \sum_{n\in E} x^n \leq \sum_{n\in\mathbb{N}} x^n$$

In addition,

$$\forall x \leq 1, \sum_{n\in E} x^n \leq \frac{1}{1-x}$$

So,

$$\limsup_{x\to 1^-} (1-x) \sum_{n\in E} x^n \leq \limsup_{x\to 1^-} (1-x) \frac{1}{1-x} = 1$$

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$$\forall E \subseteq \mathbb{N}, \limsup_{x\to 1^+} (x-1) \sum_{n\in E} \frac{1}{n^x} \leq \limsup_{x\to 1^+} (x-1) \sum_{n\in\mathbb{N}} \frac{1}{n^x}$$

$$\limsup_{x\to 1^+} (x-1) \sum_{n\in\mathbb{N}} \frac{1}{n^x} = 1$$

Indeed :

$$\int_1^{+\infty} \frac{dn}{n^x} \leq \sum_{n\in\mathbb{N}} \frac{1}{n^x} \leq 1 + \int_1^{+\infty} \frac{dn}{n^x}$$

Noticing that :

$$\int_1^{+\infty} \frac{dn}{n^x} = \left[\frac{1}{(1-x)n^{x-1}}\right]_1^{+\infty} = \frac{1}{x-1}$$

$$\limsup_{x\to+\infty} \frac{x-1}{x-1} \le \mu_3(\mathbb{N}) \le \limsup_{x\to 1^+} (x-1)(1+\frac{1}{x-1})$$

$$\mu_3(\mathbb{N}) = 1$$

Aware that $\forall E \subseteq \mathbb{N}$, $\mu_3(E) \le \mu_3(\mathbb{N})$ we obtain the desired inequality.

Lemma

Let's prove that if E has a density for μ_1 and $\mu_1(E) = l$ where $l \in [0,1]$, then E has a density for μ_2 and μ_3. Furthermore, $\mu_2(E) = \mu_3(E) = l$. .
Let $(a_n)_{n\in\mathbb{N}}$, be the characteristic function of the subset E (i.e. $a_n = 1$ if $n \in E$ and $a_n = 0$ otherwise).
Let $(b_n)_{n\in\mathbb{N}}$ be a sequence defined as follow :

$$\forall n \in \mathbb{N}, b_n = \frac{\sum_{i=1}^n a_i}{n}$$

It is obvious that $a_n = nb_n - (n-1)b_{n-1}$. Then

$$(1-x)\sum_{i\ge1} a_i x^i = (1-x)^2 \sum_{i\ge1} i b_i x^i$$

By assumption, b_n has a limit l, i.e. $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \ge N, \mid b_n - l \mid \le \epsilon$.
We can write (for $x \in [0,1]$) :

$$(1-x)^2 \mid \sum_{n>N_\epsilon} nb_n x^n - \sum_{n>N_\epsilon} l\cdot nx^n \mid \le \epsilon(1-x)^2 \sum_{n>N_\epsilon} nx^n \le \epsilon(1-x)^2 \sum_{n\ge1} nx^n$$

But $\sum_{n\ge1} nx^n = \frac{x}{(1-x)^2}$. Then

$$\begin{aligned}
\mid (1-x)\sum_{n\ge1} a_n x^n - l\cdot x \mid &= \mid (1-x)^2 \sum_{n\ge1} nb_n x^n - (1-x)^2\cdot l\cdot \sum_{n\ge1} nx^n \mid \\
&\le \mid (1-x)^2 \sum_{n>N_\epsilon} nb_n x^n - (1-x)^2 \cdot l\cdot \sum_{n>N_\epsilon} nx^n \mid \\
&\quad + \mid (1-x)^2 \sum_{n\le N_\epsilon} nb_n x^n - (1-x)^2\cdot l\cdot \sum_{n\le N_\epsilon} nx^n \mid \\
&\le \epsilon\cdot x + (1-x)^2 \mid \sum_{n\le N_\epsilon} nb_n x^n - l\cdot \sum_{n\le N_\epsilon} nx^n \mid
\end{aligned}$$

The last term in the right hand side tends to 0 when $x \to 1^-$, and we conclude that for x sufficiently close to 1, $\mid (1-x)\sum_{n\ge1} a_n x^n - l \mid \le \epsilon + \epsilon + \epsilon = 3\epsilon$. Therefore $(1-x)\sum_{n\ge1} a_n x^n$ has a limit when $x \to 1^-$, and this limit is l.

Now, let's prove that if E has a density for μ_1 then it has a density for μ_3 and $\mu_1(E) = \mu_3(E)$.
We have

$$\sum_{n\ge1} \frac{a_n}{n^x} = \sum_{n\ge1} nb_n \left(\frac{1}{n^x} - \frac{1}{(n+1)^x}\right)$$

By definition of the limit and by asumption, $\forall \epsilon > 0, \exists N_\epsilon, \forall n \geq N_\epsilon, \mid b_n - l \mid \leq \epsilon$.
Consequently,

$$\forall x > 1, \left| \sum_{n \geq N_\epsilon} n b_n \left(\frac{1}{n^x} - \frac{1}{(n+1)^x} \right) - l \sum_{n > N_\epsilon} n \left(\frac{1}{n^x} - \frac{1}{(n+1)^x} \right) \right| \leq \epsilon \sum_{n \geq 1} n \cdot \mid \frac{1}{n^x} - \frac{1}{(n+1)^x} \mid$$

The only thing we have to prove in order to conclude like previously is that $(x-1) \cdot \sum_{n \geq 1} n \cdot (\frac{1}{n^x} - \frac{1}{(n+1)^x}) \to 1$ when $x \to 1^+$. But $\sum_{n \geq 1} n \cdot (\frac{1}{n^x} - \frac{1}{(n+1)^x}) = \sum_{n \geq 1} \frac{1}{n^x}$ and the conclusion follows because $\mu_3(\mathbb{N}) = 1$.

Question 2

Let's find $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ for a finite subset E.
We have $\forall n \in \mathbb{N}, \frac{\sharp(E \cap [1,n])}{n} \leq \frac{Card(E)}{n}$.
Then : $\limsup_{n \to +\infty} \frac{\sharp(E \cap [1,n])}{n} = \mu_1(E) = 0$ where the limsup is in fact a limit. Then, according to our lemma :

$$\mu_1(E) = \mu_3(E) = 0$$

Remark : Of course it is not the most efficient way to solve the question, but we think that the lemma is interesting in itself because it is much more easy to study the density for μ_1 than for μ_2 or μ_3. We will see this in the following questions.

Question 3

Let's determine $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ for $E = \{a + nd \mid n \in \mathbb{N}\}$ and a and b fixed integers. We have $a + nd \leq N \Leftrightarrow n \leq \frac{N-a}{d}$. Therefore $\mu_1(E) = \limsup_{n \to +\infty} \frac{\frac{N-a}{d}}{N} = \frac{1}{d}$ and the limsup is in fact a limit.

And according to our lemma :

$$\mu_1(E) = \mu_2(E) = \mu_3(E) = \frac{1}{d}$$

Question 4

Let's determine $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ for $E = \{[x^n] \mid n \in \mathbb{N}\}$. If $x = 1$, E is finite and $\mu_1(E) = \mu_2(E) = \mu_3(E) = 0$. We assume $x > 1$. We have $x^n < N \Leftrightarrow n \leq \frac{\ln N}{\ln x}$ Therefore $0 \leq \mu_1(E) \leq \limsup_{N \to +\infty} \frac{\ln N}{N \ln(x)} = 0$ and the limsup is in fact a limit. By our lemma,

$$\mu_1(E) = \mu_2(E) = \mu_3(E) = 0$$

Question 6

The idea is to find a set A such which has no density for μ_1, i.e. the limsup is not a limit. By taking $B = \mathbb{N} - A$, we will show that $\mu_1(A) + \mu_1(B) > 1 = \mu_1(A \cup B)$.

Let

$$A = \bigcup_{n \geq 1} [2^{2n}; 2^{2n+1} - 1]$$

Then :

$$\sharp([1, 2^{2n+1} - 1] \cap A) = \sum_{k=1}^{n} 4^k = 4 \cdot \frac{4^n - 1}{3}$$

$$\frac{\sharp([1, 2^{2n+1} - 1] \cap A)}{2^{2n+1} - 1} = \left(4 \cdot \frac{4^n - 1}{3}\right) \left(\frac{1}{2 \cdot 4^n - 1}\right) \to \frac{2}{3}$$

when $n \to \infty$.

We have also for $n \geq 2$:

$$\sharp([1, 2^{2n} - 1] \cap A) = \sum_{k=1}^{n-1} 4^k = 4 \cdot \frac{4^{n-1} - 1}{3}$$

Hence

$$\frac{\sharp([1, 2^{2n} - 1] \cap A)}{2^{2n} - 1} = 4 \cdot \frac{4^{n-1} - 1}{3} \cdot \frac{1}{2^{2n} - 1} \to \frac{1}{3}$$

when $n \to \infty$.

Consequently

$$\limsup_{n \to +\infty} \frac{(A \cap [1, n])}{n} \geq \frac{2}{3} > \frac{1}{3} \geq \liminf_{n \to +\infty} \frac{(A \cap [1, n])}{n}$$

Let $B = \mathbb{N} - A$. We have $\mu_1(A \cup B) = 1$. Let's show that $\mu_1(A) + \mu_1(B) > 1$. We define :

$$F_A(n) = \frac{\sharp(A \cap [1, n])}{n}$$

We have :

$$F_B(n) = 1 - F_A(n)$$

$$\mu_1(B) = \limsup(1 - F_A) = 1 - \liminf F_A(n) > 1 - \limsup F_A(n) = 1 - \mu_1(a)$$

i.e.

$$\mu_1(A) + \mu_1(B) > 1$$

Then :

$$\mu_1(A) + \mu_1(B) > \mu_1(A \cup B) + \mu_1(A \cap B)$$

Question 7

We will use the same idea as previously, i.e. we will find a set A which has no density for μ_2. In fact we will take the same A as previously, i.e.

$$A = \bigcup_{n \geq 1} [2^{2n}; 2^{2n+1} - 1]$$

Let $f_A(x) = (1 - x) \sum_{n \in A} x^n$. We have :

$$f_A(x) = (1 - x) \cdot \sum_{n=1}^{+\infty} (x^{2^{2n}} + ... + x^{2^{2n+1} - 1}) = \sum_{n=1}^{+\infty} x^{2^{2n}} (1 - x^{2^{2n}})$$

We notice that

$$f_A(x^2) + f_A(x) = \sum_{k=2}^{+\infty} x^{2^k} (1 - x^{2^k}) = (1 - x) \cdot \sum_{k=2}^{+\infty} (x^{2^k} + ... + x^{2^{k+1} - 1}) = x^4 \cdot (1 - x) \cdot \sum_{n=0}^{\infty} x^n = x^4$$

Hence :

$$f_A(x) = x^4 - f_A(x^2) \text{ and } f_A(x) = x^4 - x^8 + f_A(x^4) > f_A(x^4)$$

if $0 < x < 1$. If $l = \lim_{x \to 1^-} f_A(x)$ exists, then $l = 1 - l$, so necessarily $l = \frac{1}{2}$. But if we find an $x_0 \in]0, 1[$ such that $f(x_0) > \frac{1}{2}$, then $f(x_0^{\frac{1}{4^n}}) > f(x_0) > \frac{1}{2}$, which is a contradiction if $n \to +\infty$ because $x_0^{\frac{1}{4^n}} \to 1$. It remains to find such a x_0. We can take for example $x_0 = 0.999999$ and an explicit calculation shows that $f_A(x_0) \geq \sum_{n=1}^{11} x^{2^{2n}} (1 - x^{2^{2n}}) \geq 0.5007$.

Now let $B = \mathbb{N} - A$. We have $\mu_2(A \cup B) = 1$. We will prove that $\mu_2(A \cup B) > 1$. Indeed, $F_A(x) + F_B(x) = 1$ for all $0 < x < 1$ and $\mu_2(B) = \limsup_{x \to 1^-} F_B(x) = \limsup_{x \to 1^-}(1 - F_A(x)) = 1 - \liminf_{x \to 1^-} F_A(x) > 1 - \limsup_{x \to 1^-} F_A(x) = 1 - \mu_2(A)$.

Remark : It could be interesting to answer to the question : if a set has a density for μ_2 does it has a density for μ_1 ? In other term it is the converse of our lemma. We think that it could be true under some assumptions on the set but we didn't prove it.