Problem 6 : *Recurrent Sequences*

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Abstract

All this results are for $n \geqslant 3$

1. If $u_{n+1} = \dfrac{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}{n}$, we proved that :
   - If ${u_1}^2 + {u_2}^2 = 2$: (u_n) is constant equal to 1.
   - If ${u_1}^2 + {u_2}^2 > 2$: (u_n) is increasing, admits no upper bound, and diverges to $+\infty$.
   - If ${u_1}^2 + {u_2}^2 < 2$: (u_n) is decreasing, bounded below by 0, so converges : its limit is 0.

2. If $u_{n+1} = \dfrac{u_1 \cdot u_n + u_2 \cdot u_{n-1} + \cdots + u_n \cdot u_1}{n}$, we proved that :
   - If $u_2 = 0$: (u_n) is constant equal to 0.
   - If $u_1 = 0$: for any odd n, $u_n = 0$; for any even n, u_n is positive or negative depending on the sign of u_2.
   - If $u_2 = {u_1}^2$: for any integer n, $u_n = {u_1}^n$.

1 $u_{n+1} = \dfrac{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}{n}$

Let $(u_n)_{n \geq 1}$ be a sequence of real numbers such that, for any $n \geqslant 2$,

$$u_{n+1} = \frac{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}{n}$$

Note that :

$$u_{n+1} = \frac{{u_1}^2 + {u_2}^2 + \cdots + {u_{n-1}}^2}{n} + \frac{{u_n}^2}{n} = \frac{(n-1)u_n + {u_n}^2}{n}$$

We obtain those results :

	${u_1}^2 + {u_2}^2 = 2$	${u_1}^2 + {u_2}^2 > 2$	${u_1}^2 + {u_2}^2 < 2$
monotonicity (for $n \geqslant 3$)	(u_n) is constant	(u_n) is increasing	(u_n) is decreasing
lower bound	$\min(u_1; u_2)$	$\min(u_1; u_2)$	$\min(u_1; u_2)$
upper bound	$\max(u_1; u_2; 1)$		$\max(u_1; u_2; u_3)$
convergence	(u_n) is convergent	(u_n) is divergent	(u_n) is convergent
limits	$\lim(u_n) = 1$	$\lim(u_n) = +\infty$	$\lim(u_n) = 0$

1.1 ${u_1}^2 + {u_2}^2 = 2$

We prove by mathematical induction that :

$${u_1}^2 + {u_2}^2 = 2 \Rightarrow \forall n \geqslant 3, u_n = 1$$

Basis : Show that the property holds for $n = 3$:

$$u_3 = \frac{{u_1}^2 + {u_2}^2}{2} = \frac{2}{2} = 1$$

Inductive step : Assume that $u_n = 1$. Prove that $u_{n+1} = 1$:

$$u_{n+1} = \frac{(n-1)u_n + {u_n}^2}{n} = \frac{n-1+1}{n} = 1$$

We can conclude that ${u_1}^2 + {u_2}^2 = 2 \Rightarrow \forall n \geqslant 3, u_n = 1$, hence the results of the first column.

1.2 ${u_1}^2 + {u_2}^2 \neq 2$

Similarly, we can prove by mathematical induction that :
${u_1}^2 + {u_2}^2 < 2 \Rightarrow \forall n \geqslant 3, u_n < 1$ and ${u_1}^2 + {u_2}^2 > 2 \Rightarrow \forall n \geqslant 3, u_n > 1$
Now, show that if (u_n) converges, its limit is 0 or 1.

$$u_{n+1} = \frac{1}{n}\sum_{k=1}^{n} {u_k}^2$$

Let $l = \lim (u_n)$. $\lim (u_n)^2 = l^2$.
And, by Cesàro's theorem :

$$\lim \frac{1}{n}\sum_{k=1}^{n} {u_k}^2 = l^2$$

So, $l = \lim (u_n) = \lim (u_{n+1}) = l^2 \Rightarrow l = l^2 \Rightarrow l = 1$ or $l = 0$
Calcule :

$$u_{n+1} - u_n = \frac{(n-1)u_n + {u_n}^2 - n \cdot u_n}{n} = \frac{{u_n}^2 - u_n}{n}$$

If ${u_1}^2 + {u_2}^2 > 2 : \forall n \geqslant 3, u_n > 1$, hence $u_{n+1} - u_n > 0 : (u_n)$ is strictly increasing. Its limit can't be 0 or 1 : (u_n) is divergent, its limit is $+\infty$.
If ${u_1}^2 + {u_2}^2 < 2 : \forall n \geqslant 3, u_n < 1$, hence $u_{n+1} - u_n < 0$: the sequence is strictly decreasing and bounded below by 0 : (u_n) is convergent, its limit is 0.

2 $u_{n+1} = \dfrac{u_1u_n + u_2u_{n-1} + \cdots + u_nu_1}{n}$

Let $(u_n)_{n\geq 1}$ be a sequence of real numbers such that, for any $n \geqslant 2$,

$$u_{n+1} = \frac{u_1u_n + u_2u_{n-1} + \cdots + u_nu_1}{n}$$

We can easily prove (by reasoning with mathematical induction) that : $u_2 = 0 \Rightarrow \forall n \geqslant 2, u_n = 0$. So we will consider $u_2 \neq 0$.
Similarly, we obtain :

- $(u_1; u_2) \in [-1; 1]^2 \Rightarrow \forall n \in \mathbb{N}^*, 0 \leqslant |u_n| \leqslant 1$
- $(u_1; u_2) \in]1; +\infty[^2 \Rightarrow \forall n \in \mathbb{N}^*, 1 < u_n$

2.1 $u_1 = 0$

Prove by induction that if $u_1 = 0$, then for any integer n :

$$u_{4n-3} = u_{4n-1} = 0$$
$$u_{4n} > 0$$
$$u_{4n-2} \begin{cases} > 0, & \text{if } u_2 > 0 \\ < 0, & \text{if } u_2 < 0 \end{cases}$$

Basis : for $n = 1$:

$$u_{4n-3} = u_1 = 0$$
$$u_{4n-2} = u_2 \begin{cases} > 0, & \text{if } u_2 > 0 \\ < 0, & \text{if } u_2 < 0 \end{cases}$$
$$u_{4n-1} = u_3 = \frac{u_1 \cdot u_2 + u_2 \cdot u_1}{2} = u_1 \cdot u_2 = 0$$
$$u_{4n} = u_4 = \frac{u_1 \cdot u_3 + u_2 \cdot u_2 + u_3 \cdot u_1}{3} = \frac{{u_2}^2}{3} > 0$$

Inductive step : Assume that, for any $k \leqslant n$

$$u_{4k-3} = u_{4k-1} = 0$$
$$u_{4k} > 0$$
$$u_{4k-2} \begin{cases} > 0, & \text{if } u_2 > 0 \\ < 0, & \text{if } u_2 < 0 \end{cases}$$

Prove that :

$$u_{4n+1} = u_{4n+3} = 0$$
$$u_{4n+4} > 0$$
$$u_{4n+2} \begin{cases} > 0, & \text{if } u_2 > 0 \\ < 0, & \text{if } u_2 < 0 \end{cases}$$

Calculate : $u_{4n+1} = \dfrac{1}{4n} \displaystyle\sum_{i=1}^{4n} u_i \cdot u_{4n+1-i}$. However, if i is even, $u_{4n+1-i} = 0$; and if i is odd, $u_i = 0$.

So, $u_{4n+1} = \dfrac{1}{4n} \displaystyle\sum_{i=1}^{4n} 0 = 0$. Similarly, $u_{4n+3} = 0$

And : $u_{4n+2} = \dfrac{1}{4n+1}\sum_{i=1}^{4n+1} u_i \cdot u_{4n+2-i}$

- If $u_2 > 0$:
  - If $i \equiv 1[2]$, then $u_i = u_{4n+2-i} = 0$, so $u_i \cdot u_{4n+2-i} = 0$.
  - If $i \equiv 0[4]$, then $u_i > 0$ and $u_{4n+2-i} > 0$, so $u_i \cdot u_{4n+2-i} > 0$.
  - If $i \equiv 2[4]$, then $u_i > 0$ and $u_{4n+2-i} > 0$, so $u_i \cdot u_{4n+2-i} > 0$.

  u_{4n+2} is a sum of positive numbers, so $u_{4n+2} > 0$.
- If $u_2 < 0$:
  - If $i \equiv 1[2]$, then $u_i = u_{4n+2-i} = 0$, so $u_i \cdot u_{4n+2-i} = 0$.
  - If $i \equiv 0[4]$, then $u_i > 0$ and $u_{4n+2-i} < 0$, so $u_i \cdot u_{4n+2-i} < 0$.
  - If $i \equiv 2[4]$, then $u_i < 0$ and $u_{4n+2-i} > 0$, so $u_i \cdot u_{4n+2-i} < 0$.

  u_{4n+2} is a sum of negative numbers, so $u_{4n+2} < 0$.

Similarly, u_{4n+4} is a sum of positive numbers, so $u_{4n+4} > 0$.
The property is checked.

2.2 $u_1 = -v_1$

Let (u_n) and (v_n) be two sequences such that :
$u_{n+1} = \dfrac{u_1u_n + u_2u_{n-1} + \cdots + u_nu_1}{n}$ and $v_{n+1} = \dfrac{v_1v_n + v_2v_{n-1} + \cdots + v_nv_1}{n}$.
Assume that $u_1 = -v_1$ and $u_2 = v_2$.

Prove by induction : $\forall n \in \mathbb{N}, u_{2n-1} = -v_{2n-1}, u_{2n} = v_{2n}$

Basis : for $n = 1$: $u_{2n-1} = u_1 = -v_1 = -v_{2n-1}$ and $u_{2n} = u_2 = v_2 = v_{2n}$

Inductive step : Assume that : $\forall k < n, u_{2k-1} = -v_{2k-1}, u_{2k} = v_{2k}$

Calculate :

$$u_{2n-1} = \frac{1}{2n-2}\sum_{i=1}^{2n-2} u_iu_{2n-1-i} = \frac{1}{2n-2}\sum_{i=1}^{2n-2} -v_iv_{2n-1-i} = -v_{2n-1}$$

Then :

$$u_{2n} = \frac{1}{2n-1}\sum_{i=1}^{2n-1} u_iu_{2n-i} = \frac{1}{2n-1}\sum_{i=1}^{2n-2} v_iv_{2n-1-i} = v_{2n}$$

So, we can consider $u_1 \geqslant 0$.

2.3 $u_2 = {u_1}^2$

Prove by induction : $u_2 = {u_1}^2 \Rightarrow \forall n \in \mathbb{N}, u_n = {u_1}^n$

Basis : $u_2 = {u_1}^2$

Inductive step : Assume that : $\forall k < n, u_k = {u_1}^k$. Prove : $u_n = {u_1}^n$

Calculate :

$$u_n = \frac{1}{n-1}\sum_{i=1}^{n-1} u_iu_{n-i} = \frac{1}{n-1}\sum_{i=1}^{n-1} {u_1}^{i+n-i} = \frac{(n-1){u_1}^n}{n-1} = {u_1}^n$$

So, in this case, if $u_1 < 1, \lim(u_n) = 0$. If $u_1 = 1, \lim(u_n) = 1$. If $u_1 > 1, \lim(u_n) = +\infty$

3 $u_{n+1} = \dfrac{u_1 u_{\sigma(1)} + u_2 u_{\sigma(2)} + \cdots + u_n u_{\sigma(n)}}{n}$

Let $(u_n)_{n\geq 1}$ be a sequence of real numbers and σ a random permutation such that, for any $n \geqslant 2$,

$$u_{n+1} = \frac{u_1 u_{\sigma(1)} + u_2 u_{\sigma(2)} + \cdots + u_n u_{\sigma(n)}}{n}$$

Prove by induction that if we note (v_n) the sequence studied in the first question, with $u_1 = v_1$ and $u_2 = v_2$ we have :

$$\forall n, u_n \leqslant v_n, |u_n| \leqslant |v_n|$$

Basis : According to the statement, $u_1 \leqslant v_1, u_2 \leqslant v_2, |u_1| \leqslant |v_1|$ and $|u_2| \leqslant |v_2|$.

Inductive step : Assume that : $\forall k < n, u_k \leqslant v_k, |u_k| \leqslant |v_k|$. Prove : $u_n \leqslant v_n, |u_n| \leqslant |v_n|$

Let $x(u_1, u_2, \cdots, u_n)$ and $y(u_{\sigma(1)}, u_{\sigma(2)}, \cdots, u_{\sigma(n)})$ be two vectors of $\mathbb{R}^n$.

We have $\|x\|_2 = \sqrt{\sum_{i=1}^{n} {u_i}^2} = \sqrt{\sum_{i=1}^{n} {u_{\sigma(i)}}^2} = \|y\|_2$.

With the Euclidean dot product : $\langle x; y\rangle = \sum_{i=1}^{n} u_i u_{\sigma(i)}$

For $n \geqslant 3, v_n$ is positive, so, with Cauchy-Schwarz inequality :

$$u_{n+1} \leqslant |u_{n+1}| = \frac{|\langle x; y\rangle|}{n} \leqslant \frac{\|x\|_2 \|y\|_2}{n} = \frac{{\|x\|_2}^2}{n} = \frac{{u_1}^2 + \cdots + {u_n}^2}{n} \leqslant \frac{{v_1}^2 + \cdots + {v_n}^2}{n} = v_{n+1} = |v_{n+1}|$$

So, if ${u_1}^2 + {u_2}^2 < 2$, we have : $0 \leqslant |u_n| \leqslant v_n$ and $\lim(v_n) = 0$.

With the squeeze theorem :

$${u_1}^2 + {u_2}^2 < 2 \Rightarrow \lim(u_n) = 0$$