# Problem 5 : *Stable polygons*

Descartes High School
Team TS6-Descartes

## Abstract

We denote by $P_n$ the regular polygon with $n$ verticess, $R_i$ the "reference" stable subsets with $p_i$ vertices.

We search the number of stable subsets of $P_n$.

We conjecture that **a subset is stable if and only if it is a disjoint union of "reference" stable subsets**.

Then, we prove that, **if $n$ is a prime, the only stable subset is $P_n$ himself**.

We prove this property by working on complex numbers, cyclotomic polynomials, and Einsenstein's criterion.

With binomial coefficients properties, we denombrate the number of stable subsets in different cases :

| | |
|---|---|
| if $n$ is a prime number : | the number of stable subsets is 1 |
| if $n$ is a product of two distinct prime numbers : | the number of stable subsets is $2^p + 2^q - 3$ |
| if $n$ is a power of a prime number : | the number of stable subsets is $2^{p^{m-1}} - 1$ |

# Preliminaries

## 0.1 Notations

A regular polygon is polygon whose sides and angles are equal.
All regular polygons are cyclic by definition and the center of his circumscribed circle is the isobarycenter of the polygon.

We denote by $P_n$ the regular polygon with $n$ sides.
In the whole problem we will assimilate $P_n$ to the set : $E = \{S_1; S_2; \cdots; S_n\}$ of his vertices.
We denote by $\Omega(\omega)$ his gravity center.

Let be the complex mark $(O; \overrightarrow{u}; \overrightarrow{v})$ such has $\Omega = O$ and $\overrightarrow{\Omega S_0} = \overrightarrow{u}$.
Every vertex of $P_n$ is defined by :

$$S_k \left( e^{\dfrac{2ik\pi}{p}} \right)$$

with $0 \leq k \leq p-1$ and $p$ a prime number.

## 0.2 Two preliminaries properties

### 0.2.1 If a subset $A$ is stable, his complementary $\overline{A}$ is stable too

Let be $A$ and $\overline{A}$ his complementary, subsets of $E$.
We denote by $\Omega_A$ and $\Omega_{\overline{A}}$ their respective isobarycenters.
We obtain :

$$\Omega = \text{bar}\{(\Omega_A, \sharp A); (\Omega_{\overline{A}}; \sharp \overline{A})\}$$

If $A$ is a stable subset, $\Omega_A = \Omega$.
We obtain : $\Omega = \text{bar}\{(\Omega; \sharp A); (\Omega_{\overline{A}}; \sharp \overline{A})\}$.
We obtain two different cases :

- $\Omega_{\overline{A}} = \Omega$
  So $\Omega$ is defined by : $\text{bar}\{(\Omega; \sharp A); (\Omega; \sharp \overline{A})\} = \Omega$
- $\sharp \overline{A} = 0$
  In that case, $A = E$. And we have : $\text{bar}\{(\Omega; \sharp A); (\Omega_{\overline{A}}; 0)\} = \Omega$

In all cases, we obtain :

> If $A \subset E$ is stable, his complementary $\overline{A}$ is also stable.

### 0.2.2 If $A$ and $B$ are two distinct stable subsets, $A \cup B$ is also stable.

Let be $A = \{A_1; A_2; \cdots; A_i\}$ and $B = \{B_1; B_2; \cdots; B_j\}$.
We denote by $\Omega_A$ and $\Omega_B$ their respectivel isobarycenters.
As $A$ and $B$ are stable, we have : $\Omega_A = \Omega_B = \Omega$.
We obtain :

$$\Omega_{A \cup B} = \text{bar}\{(\Omega_A; i); (\Omega_B; j)\} = \text{bar}\{(\Omega; i); (\Omega; j)\} = \Omega$$

Finally, we obtain :

> If $A$ and $B$ are two distinct stable subsets, $A \cup B$ is also stable.

## 0.3 Establishment of a conjecture.

We denote $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ his decomposition into prime numbers.
We realised an informatic program to find stable subsets for polygons from 2 to 36 vertices.
Thanks to this program, we can establish a conjecture.

Denote by $R_i$ the "reference" stable subsets with $p_i$ vertices.
We conjecture that **a subset is stable if and only if it is a disjoint union of "reference" stable subsets**.
We can take for example the polygon $P_{12}$.

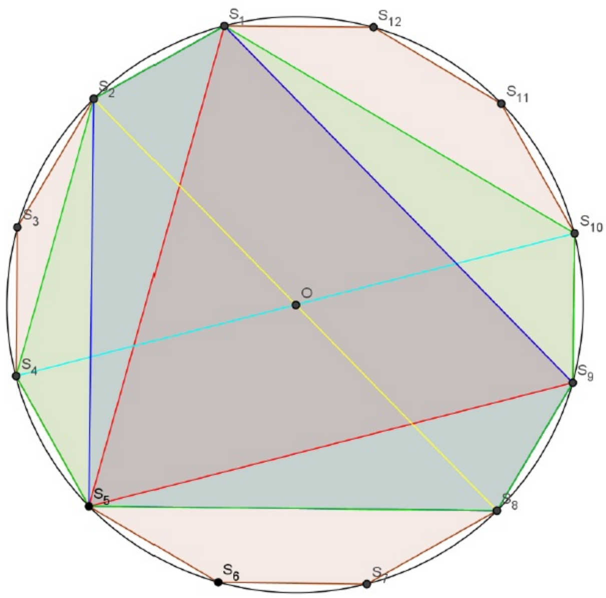



FIGURE 1 – Stable dodecagon

The triangle $S_1S_5S_8$ and the segments $[S_2S_8]$ and $[S_4S_{10}]$ are stable.
So, the pentagon $S_1S_2S_5S_8S_9$ and the heptagon $S_1S_2S_4S_5S_8S_9S_{10}$ are stable.

# 1 Number of stable subsets of $P_n$ with $n$ a prime number.

## 1.1 Observations.

Take the polygon $P_5$. We search the isobarycenters of all his subsets.

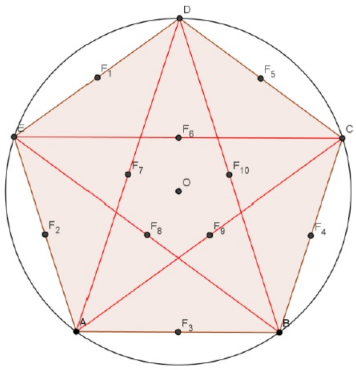



FIGURE 2 – for 2 vertices

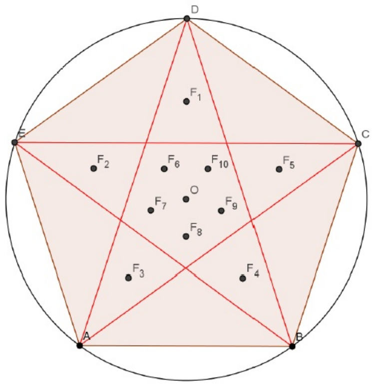



FIGURE 3 – for 3 vertices

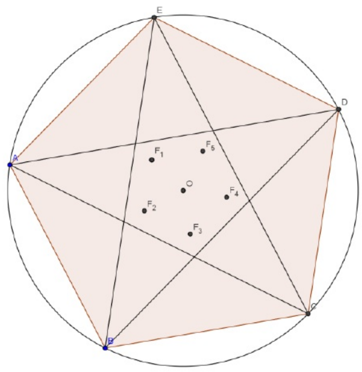



FIGURE 4 – for 4 vertices

We can see that none of these subsets are stable.
We conjecture that **the only stable subset of $P_n$ when $n$ is a prime is $P_n$ itself**.

## 1.2 Theorem 1 : the polynomial $A = 1 + z + z^2 + \cdots + z^{p-1}$ is irreductible in $\mathbb{Q}[X]$

### 1.2.1 Eisenstein's criterion

Let $A_n$ be a polynomial such as $A_n = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$.

Suppose that : $\begin{cases} p \nmid a_n \\ p \mid a_i \quad (\text{with } \ 0 \leq i \leq n-1) \\ p^2 \nmid a_0 \end{cases}$ . $A_n$ is irreductible in $\mathbb{Q}[X]$.

*Proof*

We study $A_n$ in the set $\mathbb{Z}/p\mathbb{Z}[X]$.

As $p \mid a_i \quad (\text{with } \ 0 \leq i \leq n-1)$ and $p \nmid a_n$, we obtain :

$$A_n = a_nX^n$$

Suppose that $A_n$ is reductible in $\mathbb{Q}[X]$.
The only possible solution is : $A_n = QR = a_{n_1}X^q \cdot a_{n_2}X^r$ with $p + q = n - 1$ and $a_{n_1} \cdot a_{n_2} = a_n$.
So, $p$ divides the constant term of $Q$ and $R$. We conclude that $p^2$ divides the constant term of $A_n$.

But, we initially supposed that $p^2 \nmid a_0$ with $a_0$ the constant term of $A_n$.
This is absurd. The criterion is checked.

If $\begin{cases} p \nmid a_n \\ p \mid a_i \quad (\text{with } \ 0 \leq i \leq n-1) \\ p^2 \nmid a_0 \end{cases}$ , the polynomial
$A_n = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ is irreductible in $\mathbb{Q}[X]$.

### 1.2.2 Proof of the theorem 1.

Let $A$ be the polynomial such as : $A(X) = 1 + X + X^2 + \cdots + X^{p-1}$, the $p$-th cyclotomic polynomial.
We factorise : $A(X) = \dfrac{X^p - 1}{X - 1}$

We denote $B(X) = A(X + 1)$.
We obtain :

$$B(X) = \frac{(X+1)^p - 1}{X} = \frac{\binom{p}{0}X^p + \binom{p}{1}X^{p-1} + \cdots + \binom{p}{p-1}X + \binom{p}{p} - 1}{X}$$

$$B(X) = X^{p-1} + \binom{p}{1}X^{p-2} + \cdots + \binom{p}{p-2}X + p$$

- We have $a_n = 1$ and $p \nmid a_n$.
- We have $k\binom{p}{k} = k\dfrac{p!}{(p-k)!k!} = p\dfrac{(p-1)!}{(p-1-(k-1))!(k-1)!} = p\binom{p-1}{k-1}$
  So : $p \mid k\binom{p}{k}$. But : $\gcd(p;k) = 1$. Thanks to Gauss' theorem, we obtain : $p \mid \binom{p}{k}$.
  Finally $p \mid a_i$ with $0 \leq i \leq n-1$.
- We have $a_0 = p$ and $p^2 \nmid a_0$.

Thanks to the Eiseintein's criteron, $A$ is irreductible in $\mathbb{Q}[X]$.

The polynomial $A = 1 + z + z^2 + \cdots + z^{p-1}$ is irreductible in $\mathbb{Q}[X]$.

### 1.3 Proof of the conjecture : the only stable subset is $P_n$ itself.

Let $A$ be a polynomial such as $A = 1 + z + z^2 + \cdots + z^{p-1}$. We search its roots.
$z = 1$ is obsviously not a root of $A$. Let $z \neq 1$.

We can write : $A = \dfrac{1 - z^p}{1 - z}$
So the roots of $A$ are defined by : $z^p = 1$ with $z \neq 1$.

They are all $p$-th roots, except 1. We denote by $\omega_k = e^{\frac{2ik\pi}{p}}$ with $k \neq 0$ all of its roots.
We can factorise : $A = (z - \omega_k)Q$ with $Q$ a polynomial of degree less than or equal to $p - 2$.

If there is a stable polygon with $d$ vertexes, it exists $k_0$ such as $\omega_{k_0}$ is a root of a polynomial $B = z^{\alpha_0} + z^{\alpha_1} + \cdots + z^{\alpha_{d-1}}$.
We can also factorize : $B = (z - \omega_{k_0})R$ with $R$ a polynomial of degree less than or equal to $\alpha_d - 1$.

So, $\gcd(A; B) = (z - \omega_{k_0})S$.
We obtain : $(z - \omega_{k_0}) \mid \gcd(A; B)$. As their GCD has a complex root, it is at least of degree 1. So, their GCD is not constant.
But according to the theorem 1., $A$ is irreductible in $\mathbb{Q}[X]$, that is to say, $\gcd(A; B) = \alpha \in \mathbb{Q}$ or $\gcd(A; B) = \alpha A$.

As $\gcd(A; B)$ is not constant, it is equal to $\alpha A$. So : $A \mid B$.
But, the degree of $B$ is less or equal to $A$'s one.
So, the only solution is that $A = B$.

So, we can conclude that :

There is **no stable subset**, except $P_n$ itself, when $n$ is a prime.

# 2 Number of stable subsets of $P_{pq}$ with $p$ and $q$ prime numbers.

## 2.1 Conjecture.

Let $n$ be defined by $n = pq$ with $p$ and $q$ prime numbers.
Denote $S_p$ and $S_q$ the "reference" stable subset of the polygon $P_{pq}$.
Thanks to the preliminary conjecture, we conjecture that the only stable subsets are multiples of $p$ and $q$ less than or equal to $n$.

## 2.2 Counting of the solutions.

We leave aside the case of the subset $P_n$ itself.

- Split the polygon into $p$ parts.

$$\text{We denote by : } \begin{cases} S_{(1;1)}, S_{(1;2)}, \cdots S_{(1;q)} \text{the } q \text{ vertices of } P_n \text{ included in the first } p\text{-part.} \\ S_{(2;1)}, S_{(2;2)}, \cdots S_{(2;q)} \text{the } q \text{ vertices of } P_n \text{ included in the second } p\text{-part} \\ \cdots \\ S_{(p;1)}, S_{(p;2)}, \cdots S_{(p;q)} \text{the } q \text{ vertices of } P_n \text{ included in the last } p\text{-part} \end{cases}$$

We take one vertex $S_{(1;k)}$ with $1 \leq k \leq q$ by chance. There is $\binom{q}{1}$ possible choices.
We take then all the vertices such has $S_{(l;k)}$ with $k$, always the same and $l$ from 1 to $p$.
So there is $\binom{q}{1}$ stable subsets with $p$ vertices.

We proceed in the same way for two vertices $S_{(1;k)}$. We obtain $\binom{q}{2}$ stable subsets with $2p$ vertices.
etc... until $(q-1)p$ vertices.
We obtain in this case a number :

$$\binom{q}{1} + \binom{q}{2} + \cdots + \binom{q}{q-1} = \sum_{k=1}^{q-1} \binom{q}{k}$$

- Split the polygon into $q$ parts.
  We proceed exactly in the same way.
  We obtain :

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{q-1} = \sum_{i=1}^{p-1} \binom{p}{i}$$

We add the case of $P_n$ itself.
The number of stable subset is :

$$\sum_{k=1}^{q-1} \binom{q}{k} + \sum_{i=1}^{p-1} \binom{p}{i} + 1 = \sum_{k=0}^{p} \binom{p}{k} - \binom{p}{p} - \binom{p}{0} + \sum_{i=0}^{q} \binom{q}{i} - \binom{q}{q} - \binom{q}{q} + 1 = 2^p + 2^q - 4 + 1$$

The number of stable subsets if $n = pq$ is :

$$2^p + 2^q - 3$$

# 3 Number of stable subsets of $P_{p^k}$ with $p$ a prime and $k$ an integer.

## 3.1 Conjecture.

We denote by $R_p$ the only "reference" stable subset in that case.

Thanks to the preliminary conjecture, we can conjecture that the only stable subsets are multiples of $p$ less than or equal to $p^k$.

## 3.2 Counting of the solutions.

We proceed exactly in the same way as in the part 2.2., by sharing into $p$ parts with $p^{k-1}$ vertices. We obtain :

$$\binom{p^{k-1}}{1} + \binom{p^{k-1}}{2} + \cdots + \binom{p^{k-1}}{p^{k-1}-1} = \sum_{i=1}^{p^{k-1}-1} \binom{p^{k-1}}{i}$$

$$\sum_{i=1}^{p^{k-1}-1} \binom{p^{k-1}}{i} = \sum_{i=0}^{p^{k-1}} \binom{p^{k-1}}{i} - \binom{p^{k-1}}{0} - \binom{p^{k-1}}{p^{k-1}} + 1 = 2^{p^{k-1}} - 1$$

.

The number of stable subset if $n = p^k$ is :

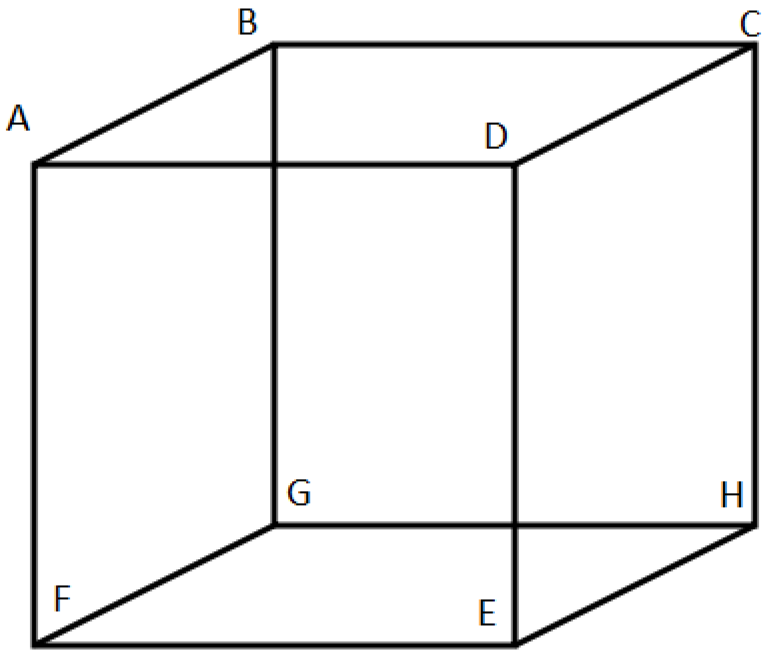
$$2^{p^{k-1}} - 1$$

# 4 Suggest and study additional directions of research.

We can for example search if there exists stable subsets of non-regular polygons and, if yes, how many.

We can also extend our research to the space.
We denote by $H_n$ a regular polyhedron with $n$ vertexes.
We can easily see that our preliminary conjecture doesn't match with a space map.

For example, with a cube $ABCDEFGH$ :




$ACEG$ is stable but not an union of stable "reference" subsets