Problem 10 : *Densities of natural subsets*

Descartes High School
Team TS6-Descartes

Abstract

Let those functionnals be "densities functionnals" of E a subset of $\mathbb{N}$:

$$\mu_1(E) = \limsup_{n\to+\infty} \frac{\sharp(E\cap[1;n])}{n}$$

$$\mu_2(E) = \limsup_{x\to 1^-}(1-x)\sum_{n\in E} x^n$$

$$\mu_3(E) = \limsup_{x\to 1^+}(x-1)\sum_{x\in E} \frac{1}{n^x}$$

- We prove first that they are well-defined.
- Then, we prove that :

if $E = \mathbb{N}$	$\mu_1(E) = \mu_2(E) = \mu_3(E) = 1$
if E is a finite subset of $\mathbb{N}$	$\mu_1(E) = \mu_2(E) = \mu_3(E) = 0$
if E is such as, for $(a,d)\in\mathbb{N}, E = \{a+dk \mid k\in\mathbb{N}\cup\{0\}\}$	$\mu_1(E) = \mu_2(E) = \mu_3(E) = \frac{1}{d}$
if E is such as, for one real $x \leq 1, E = \{[x^n] \mid n\in\mathbb{N}\cup\{0\}\}$	$\mu_1(E) = 0$
if E is the Fibonacci sequence	$\mu_1(E) = 0$
if E is the set of squares	$\mu_1(E) = 0$
if E is the set of prime numbers	$\mu_1(E) = 0$
if E is the set of numbers with at least one α in their writing in base a	$\mu_1(E) = 1$

- Finally, we prove by a counterexample, that the following property is false :

$$\mu_1(A\cup B) + \mu_1(A\cap B) = \mu_1(A) + \mu_1(B)$$

It is true if $\frac{\sharp(A\cap[1;n])}{n}$ or $\frac{\sharp(B\cap[1;n])}{n}$ has a finite limit.

Preliminaries

0.1 Notations.

$\limsup(u_n)$	superior limite of (u_n)
$\lim(u_n)$	limite of (u_n)
$\sup(u_n)$	shortest superior bound of (u_n)
$\sharp E$	cardinality of E
$[a;b]$	set of all the integers between a and b
$\mid a \mid$	absolute value of a
$[a]$	floor function of a
$u_n \sim v_n$	equivalence : $\lim \frac{u_n}{v_n} = 1$
φ	golden ratio
$\mathbb{N}$	set of all strictly positive integers
$\mathcal{P}$	set of all prime numbers with 2
$\zeta(x)$	Riemann's zeta function
$\pi(n)$	Euler's totient function

0.2 Definition of $\limsup$

We can write that : $\limsup\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} v_n$
with (v_n) be such as $v_n = \sup\limits_{m\geq n} x_m = \sup(x_n; x_{n+1}; x_{n+2}; \cdots)$.

0.3 Presentation of the density functions.

We will study the following densities :

$$\mu_1 = \limsup_{n\to+\infty} \frac{\sharp(E\cap[1;n])}{n}$$

$$\mu_2 = \limsup_{x\to1^-}(1-x)\sum_{n\in E} x^n$$

$$\mu_3 = \limsup_{x\to1^+}(x-1)\sum_{x\in E} \frac{1}{n^x}$$

0.4 Theorem 1 : If (x_n) is bounded, so $\limsup\limits_{n\to+\infty} x_n = l$ with $l \in \mathbb{R}$.

We can remark that :

- if $v_n = \sup\limits_{m\geq n} x_m = x_n$, so $v_{n+1} \leq v_n$.
- if $v_n = x_p$ with $p \geq n$, so $v_{n+1} = v_n$

Finally, (v_n) is decreasing for all (x_n).
So, if (x_n) is bounded, his superior limit is finite.
In other cases, it would be finite, or infinite.
We conclude that :

If (x_n) is bounded, $\limsup\limits_{n\to+\infty} x_n = l$ with $l \in \mathbb{R}$.

0.5 Theorem 2 : If (x_n) has a finite limit l, so $\lim\limits_{n\to+\infty} x_n = \limsup\limits_{n\to+\infty} x_n$.

If (x_n) has a finite limite l, we have : $\forall\varepsilon, \exists n_0 \mid \forall n \geq n_0, |x_n - l| < \varepsilon$.
So : $\forall\varepsilon, \exists n_0 \mid \forall n \geq n_0, |v_n - l| < \varepsilon$
Finally, $\lim x_n = \lim v_n = \limsup x_n$.

If (x_n) has a finite limit l, so $\lim\limits_{n\to+\infty} x_n = \limsup\limits_{n\to+\infty} x_n$.

0.6 Theorem 3 : If $E_1 \subset E_2$, $\mu(E_1) \leq \mu(E_2)$.

0.6.1 For μ_1

Let $E_1 \subset E_2$.
We have obviously : $\sharp(E_1 \cup [1;n]) \leq \sharp(E_2 \cup [1;n])$.
So : $\dfrac{\sharp(E_1 \cup [1;n])}{n} \leq \dfrac{\sharp(E_2 \cup [1;n])}{n}$.
Finally :

If $E_1 \subset E_2$, $\mu_1(E_1) \leq \mu_1(E_2)$

0.6.2 For μ_2

Let $E_1 \subset E_2$.
As x and n are positive, we have : $\sum\limits_{n\in E_1} x^n \leq \sum\limits_{n\in E_2} x^n$.
So : $(x-1)\sum\limits_{n\in E_1} x^n \leq (x-1)\sum\limits_{n\in E_2} x^n$
Finally :

If $E_1 \subset E_2$, $\mu_2(E_1) \leq \mu_2(E_2)$

0.6.3 For μ_3

Let $E_1 \subset E_2$.
As x and n are bigger than 1, we have : $\sum\limits_{n\in E_1} \dfrac{1}{n^x} \leq \sum\limits_{n\in E_2} \dfrac{1}{n^x}$.
So : $(x-1)\sum\limits_{n\in E_1} \dfrac{1}{n^x} \leq (x-1)\sum\limits_{n\in E_2} \dfrac{1}{n^x}$.
Finally :

If $E_1 \subset E_2$, $\mu_3(E_1) \leq \mu_3(E_2)$

1 Show that the densities are well-defined.

1.1 For μ_1

For all set E, we have :

$$0 \leq \sharp(E \cap [1;n]) \leq n$$
$$0 \leq \frac{\sharp(E \cap [1;n])}{n} \leq \frac{n}{n}$$
$$0 \leq \limsup \frac{\sharp(E \cap [1;n])}{n} \leq 1$$

As $\frac{\sharp(E \cap [1;n])}{n}$ is bounded, thanks to the theorem 1., we can conclude that $\mu_1(E)$ is finite.

Finally, $\mu_1(E)$ is well-defined.

1.2 For μ_2

For all set E we have :

$$0 \leq \sum_{n \in E} x^n \leq \sum_{n=0}^{\infty} x^n$$

But, for $0 \leq x < 1$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

We obtain : $0 \leq \sum_{n \in E} x^n \leq \frac{1}{1-x}$

$$0 \leq (1-x) \sum_{n \in E} x^n \leq 1$$

As $(1-x) \sum_{n \in E} x^n$ is bounded, thanks to the theorem 2., we can conclude that $\mu_2(E)$ is finite.

Finally, $\mu_2(E)$ is well-defined

1.3 For μ_3

Let be : $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ and $f(t) = \frac{1}{t^x}$.
We can remark that $\mu_3(\mathbb{N}) = \limsup_{x \to 1^+} (x-1)\zeta(x)$
The functionnal f is well-known and decreasing in $]0;+\infty[$
So, for $t \in [n;n+1[$, we can write :

$$\frac{1}{(n+1)^x} \leq \frac{1}{t^x} \leq \frac{1}{n^x}$$

$$\frac{1}{(n+1)^x} \leq \int_n^{n+1} \frac{1}{t^x}.dt \leq \frac{1}{n^x}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^x} \leq \int_1^{\infty} t^{-x}.dt \leq \sum_{n=1}^{\infty} \frac{1}{n^x}$$

$$\zeta(x) - 1 \leq \left[\frac{1}{-x+1} t^{-x+1}\right]_1^{+\infty} \leq \zeta(x)$$

$$\zeta(x) - 1 \leq \left[\frac{1}{(1-x)t^{x-1}}\right]_1^{+\infty} \leq \zeta(x)$$

$$\zeta(x) - 1 \leq 0 - \frac{1}{1-x} \leq \zeta(x)$$

$$(x-1)(\zeta(x)-1) \leq 1 \leq (x-1)\zeta(x)$$

So, we have on the one hand :

$$1 \leq (x-1)\zeta(x)$$

And on the other hand :

$$(x-1)\zeta(x) - (x-1) \leq 1$$

$$(x-1)\zeta(x) \leq x$$

So, we can write :

$$1 \leq (x-1)\zeta(x) \leq x$$

Finally :

$$1 \leq \limsup_{x \to 1^+} (x-1)\zeta(x) \leq 1$$

We can conclude that $\mu_3(\mathbb{N}) = 1$. So, thanks to the theorem 3., $\mu_3(E) \leq 1$.
Furthermore, as n and x are positive, μ_3 is obviously positive.
So, we have $0 \leq \mu_3(E) \leq 1$.

Finally, μ_3 is well-defined.

2 Densities of a finite set.

Let E be a finite set of $\mathbb{N}$.

2.1 For μ_1

As E is finite, ite has a biggest element n_0.
So $0 \leq \frac{\sharp(E \cap [1;n])}{n} \leq \frac{n_0}{n}$. But $\lim_{n\to+\infty} \frac{n_0}{n} = 0$.
With the theorem, we can conlude that :

If E is a finite set, $\mu_1(E) = 0$

2.2 For μ_2

As E is finite, he has a biggest element n_0.

So $0 \leq \sum_{n\in E} x^n \leq \sum_{n=0}^{n_0} x^n$

$0 \leq (1-x)\sum_{n\in E} x^n \leq (1-x)\sum_{n=0}^{n_0} x^n$

But we have : $(1-x)\sum_{n=0}^{n_0} x^n = (1-x)\frac{1-x^{n+1}}{1-x} = 1-x^{n+1}$.

And $\lim_{x\to 1^-} 1-x^{n+1} = 0$.
We conclude that :

If E is a finite set, $\mu_2(E) = 0$

2.3 For μ_3

As x and n are bigger than 1, we can write : $0 \leq \frac{1}{n^x} \leq 1$.
So, as E is a finite set, we obtain :

$$0 \leq \sum_{n\in E} \frac{1}{n^x} \leq \sharp E$$

$$0 \leq (x-1)\sum_{n\in E} \frac{1}{n^x} \leq (x-1)\sharp E$$

But, $\limsup_{x\to 1^+} (x-1)\sharp E = 0$
We obtain :

$$\limsup_{x\to 1^+} (x-1)\sum_{n\in E} \frac{1}{n^x} = 0$$

We conclude that :

If E is a finite set, $\mu_3(E) = 0$

3 Densities of an arithmetic progression.

Let E_0 be defined by : $E_0 = \{a + dk \mid (a, d, k) \in \mathbb{N}^3\} \cap [1; n]$
And E_1 defined by : $E_1 = \{a + 1 + dk \mid (a, d, k) \in \mathbb{N}^3\} \cap [1; n]$
And E_2 defined bys : $E_2 = \{a + 2 + dk \mid (a, d, k) \in \mathbb{N}^3\} \cap [1; n]$
...
For all E_i with i from 0 to $d - 1$.

3.1 For μ_1

We have : $\sharp[0; a - 1] + \sharp E_0 + \sharp E_1 + \cdots + \sharp E_{d-1} = n$
So : $\sharp E_0 + \sharp E_1 + \cdots + \sharp E_{d-1} = n - a$

But : $\forall i \in [1; d - 1], \sharp E_i \leq \sharp E_0$
So :

$$\sharp E_0 + \sharp E_1 + \cdots + \sharp E_{d-1} \leq d \cdot \sharp E_0 \leq n$$

$$n - a \leq d \cdot \sharp E_0 \leq n$$

$$\frac{n-a}{d} \leq \sharp E_0 \leq \frac{n}{d}$$

$$\frac{n-a}{dn} \leq \frac{\sharp E_0}{n} \leq \frac{n}{dn}$$

$$\frac{1 - \frac{a}{n}}{d} \leq \frac{\sharp E_0}{n} \leq \frac{1}{d}$$

But $\lim\limits_{n \to +\infty} \frac{1 - \frac{a}{n}}{d} = \frac{1}{d}$
So : $\mu_1(E_0) = \frac{1}{d}$.
We can conclude that :

If $E = \{a + dk \mid (a, d, k) \in \mathbb{N}^3\}$, $\mu_1(E) = \frac{1}{d}$

3.2 For μ_2

For $n \to +\infty$, we can write that :

$$\sum_{n=0}^{a-1} x^n + \sum_{n \in E_0} x^n + \sum_{n \in E_1} x^n + \cdots + \sum_{n \in E_{d-1}} x^n = \sum_{n=0}^{\infty} x^n$$

$$\sum_{n \in E_0} x^n + \sum_{n \in E_1} x^n + \cdots + \sum_{n \in E_{d-1}} x^n = \frac{1}{1-x} - \frac{1-x^a}{1-x}$$

$$\sum_{n \in E_0} x^n + \sum_{n \in E_1} x^n + \cdots + \sum_{n \in E_{d-1}} x^n = \frac{x^a}{1-x}$$

But :

$$\forall i \in [1; d-1], \sum_{n \in E_i} x^n \leq \sum_{n \in E_0} x^n$$

We obtain :

$$\sum_{n \in E_0} x^n + \sum_{n \in E_1} x^n + \cdots + \sum_{n \in E_{d-1}} x^n \leq d \cdot \sum_{n \in E_0} x^n \leq \sum_{n=0}^{\infty} x^n$$

$$\frac{x^a}{1-x} \leq d \cdot \sum_{n \in E_0} x^n \leq \frac{1}{1-x}$$

$$\frac{x^a}{d} \leq (1-x) \sum_{n \in E_0} x^n \leq \frac{1}{d}$$

But $\lim_{x \to 1^-} \frac{x^a}{d} = \frac{1}{d}$

So $\mu_2(E_0) = \frac{1}{d}$

We can conclude that :

$$\text{If } E = \{a + dk \mid (a, d, k) \in \mathbb{N}^3\}, \ \mu_2(E) = \frac{1}{d}$$

3.3 For μ_3

As $a > 0$, we can write :

$$\frac{1}{(a+d)^x} + \frac{1}{(a+2d)^x} + \frac{1}{(a+3d)^x} + \cdots \leq \frac{1}{d^x} + \frac{1}{(2d)^x} + \frac{1}{(3d)^x} + \cdots$$

$$\sum_{n \in E_0} \frac{1}{n^x} \leq \sum_{k=1}^{+\infty} \frac{1}{(kd)^x}$$

$$\sum_{n \in E_0} \frac{1}{n^x} \leq \frac{1}{d^x} \sum_{k=1}^{+\infty} \frac{1}{k^x}$$

$$\sum_{n \in E_0} \frac{1}{n^x} \leq \frac{1}{d^x} \zeta(x)$$

But, we proved in section 1.3. that : $1 \leq (x-1)\zeta(x) \leq x$.

We obtain :

$$\frac{1}{d^x} \leq (x-1) \sum_{n \in E_0} \frac{1}{n^x} \leq (x-1) \frac{1}{d^x} \zeta(x) \leq \frac{x}{d^x}$$

$$\limsup_{x \to 1^+} (x-1) \sum_{n \in E_0} \frac{1}{n^x} = \frac{1}{d}$$

So, we conclude :

$$\text{If } E = \{a + dk \mid (a, d, k) \in \mathbb{N}^3\}, \ \mu_3(E) = \frac{1}{d}$$

4 Densities of a set : $E = \{[x^k] \mid k \in \mathbb{N}\}$

4.1 For μ_1

- ***First, we will study the case where x is an integer bigger than or equal to 1.***

  Let k be such as : $k = \sharp(E \cap [1;n])$.
  So $x^k \in \{E \cap [1;n]\}$.
- If $x = 1$, the set is such as : $E = \{1\}$. So, as it is finite, thanks to part 2., $\mu_1(E) = \mu_2(E) = \mu_3(E) = 0$.
- If $x > 1$, we obtain :

$$1 \leq x^k \leq n$$

$$0 \leq k \cdot \ln x \leq \ln n$$

$$0 \leq k \leq \frac{\ln n}{\ln x}$$

$$0 \leq \frac{\sharp(E \cap [1;n])}{n} \leq \frac{\ln n}{n \cdot \ln x}$$

But $\lim\limits_{n\to+\infty} \frac{1}{\ln x} \cdot \frac{\ln n}{n} = 0$
We obtain :

$$\lim_{n\to+\infty} \frac{\sharp(E \cap [1;n])}{n} = 0$$

Finally, thanks to theorem 2. :

For an integer x bigger or equal to 1, if $E = \{x^k \mid k \in \mathbb{N}\}$, $\mu_1(E) = 0$.

- ***Study now the general case : $x \geq 1$***

We have the inequality :

$$[x] \leq x < [x] + 1$$

$$[x]^n \leq x^n < ([x] + 1)^n$$

$$[[x]^n] \leq [x^n] \leq [([x] + 1)^n]$$

$$[x]^n \leq [x^n] \leq ([x] + 1)^n$$

But, as $[x], [x]+1 \in \mathbb{N}^2$, we have $\mu_1([x]^n) = 0$ and $\mu_1(([x]+1)^n) = 0$ thanks to the last paragraph. So, we conclude that :

For $x \geq 1$, if $E = \{[x^k] \mid k \in \mathbb{N}\}$, $\mu_1(E) = 0$.

5 Densities of other subsets of $\mathbb{N}$

5.1 Study of Fibonacci sequence

We denote by F the Fibonacci sequence.
First, prove that : $F \sim \dfrac{\varphi^n}{\sqrt{5}}$

By definition, we have : $F : F_{n+2} = F_{n+1} + F_n$
We write its characteristic equation : $x^2 - x - 1 = 0$.

Its discriminant is : $\Delta = 5$.
So the solutions are :

$$x_1 = \frac{1+\sqrt{5}}{2} = \varphi \quad x_2 = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}$$

with φ the golden ratio.

Thanks to the property, we obtain :

$$F_n = \alpha x_1^n + \beta x_2^n = \alpha\varphi^n - \beta \cdot \frac{1}{\varphi}$$

Now determinate α and β.
We have, by definition of Fibonacci sequence : $F_0 = 0$ and $F_1 = 1$.
So we solve :

$$\begin{cases} \alpha + \beta = 0 \\ \alpha\varphi - \beta \cdot \dfrac{1}{\varphi} = 1 \end{cases}$$

$$\begin{cases} \beta = -\alpha \\ \alpha\left(\varphi + \dfrac{1}{\varphi}\right) = 1 \end{cases}$$

$$\begin{cases} \beta = -\alpha \\ \alpha = \dfrac{\varphi}{\varphi^2 + 1} \end{cases}$$

$$\begin{cases} \alpha = \dfrac{1+\sqrt{5}}{5+\sqrt{5}} = \dfrac{1}{\sqrt{5}} \\ \beta = -\dfrac{1}{\sqrt{5}} \end{cases}$$

We obtain the formula, known as "Binet's formula" :

$$F_n = \frac{\varphi^n - \dfrac{1}{\varphi^n}}{\sqrt{5}}$$

So :

$$\frac{F_n \cdot \sqrt{5}}{\varphi^n} = 1 - \frac{1}{\varphi^{2n}}$$

But : $\lim\limits_{n\to+\infty} 1 - \dfrac{1}{\varphi^{2n}} = 1$.

We conclude that :

$$F_n \sim \frac{\varphi^n}{\sqrt{5}}$$

Then, let us calculate $\mu_1(F_n)$

As $F_n \sim \frac{\varphi^n}{\sqrt{5}}$, we have : $\lim_{n\to+\infty} F_n = \lim_{n\to+\infty} \frac{\varphi^n}{\sqrt{5}}$.

We write : $k = \sharp(F \cap [1;n])$. So, $F_k \in \{F \cap [1;n]\}$.
We obtain :

$$\frac{\varphi^k}{\sqrt{5}} \leq n$$

$$\varphi^k \leq n\sqrt{5}$$

$$k \cdot \ln\varphi \leq \ln(n\sqrt{5})$$

$$k \leq \frac{\ln(n\sqrt{5})}{\ln\varphi}$$

Thanks to the theorem 2., we obtain :

$$\limsup_{n\to+\infty} \frac{\sharp(F \cap [1;n])}{n} = \lim_{n\to+\infty} \frac{\ln(n\sqrt{5})}{\ln\varphi \cdot n} = 0$$

If F is the Fibonacci sequence, we have $\mu_1(F) = 0$.

5.2 Study of the set of squares.

Let S be the set of all perfect squares in $\mathbb{N}$.
So, we have : $\sharp(S \cap [1;n]) = [\sqrt{n}]$.
We obtain : $0 \leq \frac{\sharp(S \cap [1;n])}{n} = \frac{[\sqrt{n}]}{n} \leq \frac{1}{\sqrt{n}}$.
But, $\lim_{n\to+\infty} \frac{1}{\sqrt{n}} = \limsup_{n\to+\infty} \frac{1}{\sqrt{n}} = 0$.
So, we conclude that :

If S is the set of all perfect squares, we have $\mu_1(S) = 0$.

5.3 Study of $\mathcal{P}$

Let $\mathcal{P}$ be the set of prime numbers.
We have : $\sharp(\mathcal{P} \cap [1;n]) = \pi(n)$ with $\pi(n)$ Euler's totient function.

We know that :

$$\pi(n) \sim \frac{n}{\log n}$$

So, we calculate :

$$\mu_1(\mathcal{P}) = \lim_{n\to+\infty} \frac{\sharp(\mathcal{P} \cap [1;n])}{n} = \lim_{n\to+\infty} \frac{\pi(n)}{n} = \lim_{n\to+\infty} \frac{1}{\log(n)} = 0$$

So, we conclude :

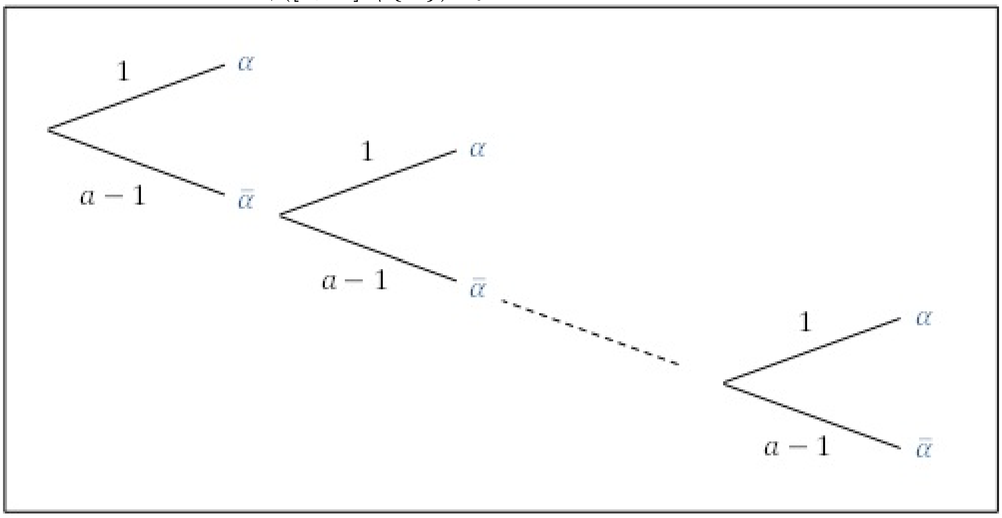
> If $\mathcal{P}$ is the set of prime numbers, we have $\mu_1(\mathcal{P}) = 0$.

5.4 Study of the set of numbers with at least one α in their writing in basis a.

Let A be the set of numbers with at least one α such as $0 \leq \alpha < a$ in their writing in basis $a \geq 2$.
We denote $N = a^n - 1$.
Let us calculate $\limsup_{N\to+\infty} \frac{\sharp(A \cap [0;N])}{N}$.

We have : $\sharp(A \cap [0;N]) = \sharp[0;N] - \sharp([0;N] \setminus \{A\})$

We find the value of $\sharp([0;N] \setminus \{A\})$ by a tree :



So, we have : $\sharp([0;N] \setminus \{A\}) = (\sharp\overline{\alpha})^n = (a-1)^n$
And : $\sharp(A \cap [0;N]) = a^n - (a-1)^n$.

We obtain :

$$\limsup_{N\to+\infty} \frac{\sharp(A \cap [0;N])}{N} = \limsup_{N\to+\infty} \frac{a^n - (a-1)^n}{a^n - 1} = \limsup_{N\to+\infty} 1 - \frac{(a-1)^n}{a^n - 1} = 1$$

But, as A is a subset of $\mathbb{N}$ and $A \cap [0;n]$ of $E \cap [1;n]$ we have :

$$\limsup_{N\to+\infty} \frac{\sharp(E \cap [0;N])}{N} \leq \limsup_{N\to+\infty} \frac{\sharp(E \cap [1;n])}{n} \leq 1$$

So, we conclude :

> If A is the set of numbers with at least one α in their writing in basis $a \geq 2$
> $\mu_1(A) = 1$.

6 Is it true that : $\mu_1(A \cup B) + \mu_1(A \cap B) = \mu_1(A) + \mu_1(B)$?

6.1 Necessary condition.

We have, obviously, the equality :

$$\sharp((A \cup B) \cap [1;n]) = \sharp(A \cap [1;n]) + \sharp(B \cap [1;n]) - \sharp((A \cap B) \cap [1;n])$$

$$\frac{\sharp((A \cup B) \cap [1;n])}{n} + \frac{\sharp((A \cap B) \cap [1;n])}{n} = \frac{\sharp(A \cap [1;n])}{n} + \frac{\sharp(B \cap [1;n])}{n}$$

Now we have two cases :

6.1.1 If $\frac{\sharp(A \cap [1;n])}{n}$ or $\frac{\sharp(B \cap [1;n])}{n}$ have a finite limit.

In this case, thanks to the theorem 2,we have : $\lim\limits_{n \to +\infty} \frac{\sharp(A \cap [1;n])}{n} = \limsup\limits_{n \to +\infty} \frac{\sharp(A \cap [1;n])}{n}$ and similarly for B.

So, we obtain :

$$\lim_{n \to +\infty} \left(\frac{\sharp((A \cup B) \cap [1;n])}{n} + \frac{\sharp((A \cap B) \cap [1;n])}{n} \right) = \lim_{n \to +\infty} \left(\frac{\sharp(A \cap [1;n])}{n} + \frac{\sharp(B \cap [1;n])}{n} \right)$$

$$\lim_{n \to +\infty} \frac{\sharp((A \cup B) \cap [1;n])}{n} + \lim_{n \to +\infty} \frac{\sharp((A \cap B) \cap [1;n])}{n} = \lim_{n \to +\infty} \frac{\sharp(A \cap [1;n])}{n} + \lim_{n \to +\infty} \frac{\sharp(B \cap [1;n])}{n}$$

$$\limsup_{n \to +\infty} \frac{\sharp((A \cup B) \cap [1;n])}{n} + \limsup_{n \to +\infty} \frac{\sharp((A \cap B) \cap [1;n])}{n} = \limsup_{n \to +\infty} \frac{\sharp(A \cap [1;n])}{n} + \limsup_{n \to +\infty} \frac{\sharp(B \cap [1;n])}{n}$$

$$\mu_1(A \cap B) + \mu_1(A \cup B) = \mu_1(A) + \mu_2(B)$$

We conclude that the property is checked if $\lim\limits_{n \to +\infty} \frac{\sharp(A \cap [1;n])}{n} = \limsup\limits_{n \to +\infty} \frac{\sharp(A \cap [1;n])}{n}$ and similarly for B.

6.1.2 If $\frac{\sharp(A \cap [1;n])}{n}$ and $\frac{\sharp(B \cap [1;n])}{n}$ have not a finite limit.

As $\frac{\sharp(A \cap [1;n])}{n}$ and $\frac{\sharp(B \cap [1;n])}{n}$ are bounded between 0 and 1, the only possibility for them to not have a finite limite, is to not have limit at all.

As, $\limsup(u_n + v_n) \neq \limsup u_n + \limsup v_n$, we can't conclude.

We can see this relation in a simple example.

Let $u_n = \left\{0; \frac{1}{2}; 0; \frac{2}{3}; 0; \frac{3}{4}; \cdots \right\}$ and $v_n = \left\{\frac{1}{2}; 0; \frac{2}{3}; 0; \frac{3}{4}; 0; \cdots \right\}$.

We have : $\limsup\limits_{n \to +\infty} u_n = \limsup\limits_{n \to +\infty} v_n = 1$

But $u_n + v_n = \left\{\frac{1}{2}; \frac{1}{2}; \frac{2}{3}; \frac{2}{3}; \frac{3}{4}; \frac{3}{4}; \cdots\right\}$.
And we have $\limsup_{n\to+\infty}(u_n + v_n) = 1$.

So we have : $\limsup_{n\to+\infty}(u_n + v_n) \neq \limsup_{n\to+\infty} u_n + \limsup_{n\to+\infty} v_n$.

So, we must search if $\frac{\sharp(A \cap [1;n])}{n}$ can have no limit. If it can, the property can be false.

6.2 Counterexample.

Let A be the set of integers which begins with a 1.
We have for all n :

$$\frac{\sharp(A \cap [1; 10^n - 1])}{10^n - 1} = \frac{1}{9}$$

because $\frac{1}{9}$ of numbers between 1 and $999\cdots999$ begin with a 1.

$$\lim_{n\to+\infty} \frac{\sharp(A \cap [1; 2 \cdot 10^n - 1])}{2 \cdot 10^n - 1} = \frac{1}{9} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{5}{9} = \frac{30}{54}$$

because all numbers between $100\cdots000$ and $199\cdots999$ begin with a 1.

So, $\frac{\sharp(A \cap [1;n])}{n}$ has no finite limit and we have :

$$\mu_1(A) = \limsup_{n\to+\infty} \frac{\sharp(A \cap [1;n])}{n} = \frac{30}{54}$$

Let B be the set of integers which begin with a 2.
We have for all n :

$$\frac{\sharp(B \cap [1; 2 \cdot 10^n - 1])}{2 \cdot 10^n - 1} = \frac{1}{9} \cdot \frac{1}{2} + 0 = \frac{1}{18} = \frac{3}{54}$$

because $\frac{1}{9}$ of numbers between 1 and $999\cdots999$ begin with a 2 and 0 between $100\cdots000$ ans $199\cdots999$.

$$\lim_{n\to+\infty} \frac{\sharp(A \cap [1; 3 \cdot 10^n - 1])}{3 \cdot 10^n - 1} = \frac{1}{9} \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{3} = \frac{10}{27} = \frac{20}{54}$$

because all numbers between $200\cdots000$ and $299\cdots999$ begin with a 2.

So, $\frac{\sharp(B \cap [1;n])}{n}$ has no finit limite and we have :

$$\mu_1(B) = \limsup_{n\to+\infty} \frac{\sharp(B \cap [1;n])}{n} = \frac{20}{54}$$

We also have, as $A \cap B = \emptyset$:

$$\mu_1(A \cap B) = 0$$

And :

$$\lim_{n\to+\infty} \frac{\sharp((A\cup B)\cap[1;3\cdot 10^n-1])}{3\cdot 10^n-1} = \frac{2}{9}\cdot\frac{1}{3}+1\cdot\frac{1}{3}+1\cdot\frac{1}{3} = \frac{20}{27} = \frac{40}{54}$$

because all $\frac{2}{9}$ of numbers begin with a 1 or a 2 between 1 and $999\cdots999$; and all numbers begin with a 1 or a 2 between $100\cdots000$ and $299\cdots999$.

So, we obtain :

$$\mu_1(A\cup B) = \frac{40}{54}$$

But, we have :

$$\mu_1(A)+\mu_1(B) = \frac{30}{54}+\frac{20}{54} = \frac{50}{54}$$

$$\mu_1(A\cap B)+\mu_1(A\cup B) = 0+\frac{40}{54} = \frac{40}{54}$$

.

So, the formula $\mu_1(A)+\mu_1(B) = \mu_1(A\cap B)+\mu_1(A\cup B)$ is false in this case.

6.3 Conclusion.

Finally, we obtain :

The formula $\mu_1(A)+\mu_1(B) = \mu_1(A\cap B)+\mu_1(A\cup B)$ is false in a general case.

It is trueif $\frac{\sharp(A\cap[1;n])}{n}$ or $\frac{\sharp(B\cap[1;n])}{n}$ has a finite limit.