Problem 1: *Generalizing Perfectness*

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Abstract

We study f-perfect integers for various functions f.

1. With $f(n) = \tau(n)$ we prove by double implication that for all $n \geq 1$ n is τ-perfect if and only if n is a square of a prime.

   If $f(n) = \tau(n) + k$ we prove that if $k \geq 1$ then there is no f-perfect integer.

2. With $f(n) = \varphi(n)$ we show that if $n = 2^\alpha$ then n is φ-perfect. The reverse stays open.

3. For $f(n) = n + a$ we show that if $n = 2^k P$ with P a prime and such as $P = 2^{k+1} + 2ak - 1$ then n is f-perfect. For $f(n) = an$ we show that the integers which are perfect are also f-perfect. If $f(n) = a(n + b)$ we show that a integer n is f-perfect if $n = 2^k P$ with P a prime and such as $P = 2^{k+1} + 2bk - 1$.

4. For $f(n) = \ln(n)$ we show that the integers ln-perfect are the products of 2 different prime numbers and the cubes of prime numbers.

5. for $f(n) = (-1)^n$ we show that n is f-perfect if and only if n is an odd prime or n is such as $4p$ with p a prime.

6. For $f(n) = \binom{m}{n}$ we show that if $m - 1$ is prime then $n = m - 1$ is f-perfect. We suppose that if $m - 1$ is not a prime then there is no f-perfect integer.

   So for the case $f(n) = \binom{2012}{n}$ we have 2011 which is f-perfect because it is a prime.

7. For the other functions we have : for $f(n) = \frac{1}{n}$, $f(n) = a^n$ and $f(n) = n^a$ we show that there is no f-perfect integer.

8. For f-amicable integers : for $f(n) = \frac{1}{n}$; $f(n) = n^a$; $f(n) = a^n$ with $a \geq 2$ and $f(n) = \ln(n)$ we show that there is no f-amicable pairs. For $f(n) = (-1)^n$ we show that the f-amicable pairs are such as (p_1, p_2) with p_1 and p_2 are odd prime integers or such as $(4p_1, 4p_2)$ with p_1 and p_2 prime integers. For $f(n) = \tau(n)$ we show that every pair such as (p_1^2, p_2^2) with p_1 and p_2 prime integer is f-amicable. For $f(n) = k$ with $k \neq 0$ we show that the f-amicable pairs are such as (p_1, p_2) with $(p_1, p_2) \in P^2$.

   For $f(n) = a^n$ and $f(n) = n^a$ we show that there is no f-perfect integer.

Preliminaries

0.1 Notations

$\mathbb{N} = \{0; 1; 2; 3; \ldots\}$	set of natural numbers with 0
$\mathbb{R}, \mathbb{R}^2$	real line and real plane
$D(n) = \{d_1; d_2; \ldots; d_{\tau(n)}\}$	the set of the divisors of n
$\tau(n)$	number of natural divisors of n
$\varphi(n)$	Euler's totient function
$\mu(n)$	Moebius function

0.2 Definitions

1. Definition of a perfect number:

   We say that a integer n is perfect if:

   $$n = \sum_{\substack{d \mid n \\ 1 \leq d < n}} d$$

   Which is, of course, equivalent to:

   $$2n = \sum_{d \mid n} d$$

   Euclide, in his famous book *The Elements*, proved that if $2^k - 1$ is prime then $2^{k-1}(2^k - 1)$ is a perfect integer. Few centuries later, Euler proved that the reverse was true: if n is an even perfect number then $n = 2^{k-1}(2^k - 1)$ with $2^k - 1$ a prime.

   Nowadays, we still ignore if there are odd perfect numbers.

2. Definition of an f-perfect integer if f is arithmetic:

   $$f(n) = \sum_{\substack{d \mid n \\ 1 \leq d < n}} f(d)$$

3. We also call back that n is f-deficient if:

   $$f(n) > \sum_{\substack{d \mid n \\ 1 \leq d < n}} f(d)$$

4. And n is f-abundant if:

   $$f(n) < \sum_{\substack{d \mid n \\ 1 \leq d < n}} f(d)$$

5. We call back that for all $n \geq 2$ there is an unique decomposition in prime factors:

   $$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

6. In our report we do not study the case where $n = 1$ because the sum is not defined in this case and consequently n is never f-perfect.

0.3 Properties

1. We have

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

2. Consequently we obtain the number of divisors of n with:

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$$

3. Show that if m is f and g-perfect then m is h-perfect for $h(n) = \alpha \cdot f(n) + \beta \cdot g(n)$ with $(\alpha, \beta) \in \mathbb{R}^2$

   We have:

$$f(m) = \sum_{\substack{d|m \\ 1 \leq d < m}} f(d)$$

   and

$$g(m) = \sum_{\substack{d|m \\ 1 \leq d < m}} g(d)$$

   Calculate:

$$\begin{aligned} h(m) = \alpha \cdot f(m) + \beta \cdot g(m) &= \alpha \sum_{\substack{d|m \\ 1 \leq d < m}} f(d) + \beta \sum_{\substack{d|m \\ 1 \leq d < m}} g(d) \\ &= \sum_{\substack{d|m \\ 1 \leq d < m}} \alpha \cdot f(d) + \sum_{\substack{d|m \\ 1 \leq d < m}} \beta \cdot g(d) \\ &= \sum_{\substack{d|m \\ 1 \leq d < m}} h(d) \end{aligned}$$

So m is h-perfect for any combination of f and g.

1 $f(n) = \tau(n)$

1.1 Show that an integer $n \geq 1$ is τ-perfect if and only if n is the square of a prime.

1.1.1 Direct way:

Let $n = p^2$ with p a prime. We have $D(n) = \{1, p, p^2\}$. So $\tau(n) = 3$

Calculate:

$$\sum_{d|n} d = \tau(1) + \tau(p) + \tau(p^2) = 1 + 2 + 3 = 6 = 2\tau(n)$$

We have effectivly n f-perfect if n the square of a prime.

1.1.2 Indirect way:

We know that n is τ-perfect and $D(n) = \{d_1 = 1, d_2, \ldots, d_{\tau(n)} = n\}$ and $\tau(n) \geq 2$ if $n \geq 2$.

So:

$$\begin{aligned}
&\tau(n) = \tau(1) + \tau(d_2) + \cdots + \tau(d_{\tau(n)-1}) \\
&\tau(n) \geq 1 + 2(\tau(n) - 2) \\
&\tau(n) \leq 3
\end{aligned}$$

So we have 3 cases to study:

- If $N = 1$: we have $n = 1$, which give us a contradiction because the sum is not defined.
- If $\tau(n) = 2$, we have n a prime. But:

  $$\sum_{\substack{d|n \\ 1 \leq d < n}} \tau(d) = \tau(1) = 1 \neq 2$$

  So there is a contradiction; it is impossible.
- If $\tau(n) = 3$, we have consequently $\prod(\alpha_i + 1) = 3$. The only one possiblity is one $\alpha_i = 2$. So we have n the square of a prime p.

  This is checked for $N = 3$. So if n is f-perfect then n is the square of a prime.

Consequently we shew by double implication that an integer $n \geq 1$ is τ-perfect if and only if is the square of a prime.

1.2 This time we have to find all the f-perfect integers $n \geq 1$ with $f(n) = \tau(n) - k$. Let's try with some values of k.

1.2.1 For $f(n) = \tau(n) - 1$

We want :

$$\tau(n) - 1 = \sum_{\substack{d|n \\ 1 \leq d < n}} \tau(d) - 1$$

$$\tau(n) - 1 = \sum_{\substack{d|n \\ 1 \leq d < n}} \tau(d) - (\tau(n) - 1)$$

$$2\tau(n) - 2 = \sum_{\substack{d|n \\ 1 \leq d < n}} \tau(d)$$

Study three cases:

- If n is divisible by at least three distinct primes : $n = kp^{\alpha}q^{\beta}r^{\gamma}$ with p, q, r prime numbers and k an integer.

  We have : $kqr|n$; $kpr|n$; $kpq|n$; $kp|n$; $kq|n$. We obtain :

  $\tau(kqr) \geq \frac{\tau(n)}{2}$

  $\tau(kpr) \geq \frac{\tau(n)}{2}$

  $\tau(kpq) \geq \frac{\tau(n)}{2}$

  $\tau(kp) \geq \frac{\tau(n)}{4}$

  $\tau(kq) \geq \frac{\tau(n)}{4}$

  Hence $\sum_{\substack{d|n \\ 1 \leq d < n}} \tau(d) \geq 2\tau(n) > 2\tau(n) - 2$: n is not f-perfect.

- If $n = p^{\alpha}q^{\beta}$

  1. If $\alpha \geq 2$ and $\beta \geq 2$, we obtain :
     $\tau(p^{\alpha-1}q^{\beta}) \geq \frac{2\tau(n)}{3}$
     $\tau(p^{\alpha}q^{\beta-1}) \geq \frac{2\tau(n)}{3}$
     $\tau(p^{\alpha-2}q^{\beta}) \geq \frac{\tau(n)}{3}$
     $\tau(p^{\alpha}q^{\beta-2}) \geq \frac{\tau(n)}{3}$

     Hence $\sum_{\substack{d|n \\ 1 \leq d < n}} \tau(d) \geq 2\tau(n) > 2\tau(n) - 2$: n is not f-perfect.

2. If $\beta = 1$ (or $\alpha = 1$, without less of generality), $n = p^\alpha q$.
   $D(n) = \{1; p; p^2; \ldots; p^\alpha; q; pq; p^2q; \ldots; p^\alpha q\}$
   $f(n) = 2\alpha + 1$
   $\sum_{\substack{d|n \\ 1\leq d<n}} f(d) = 0 + 1 + 2 + \ldots + \alpha + 1 + 3 + \ldots + 2\alpha - 1$
   $= \frac{\alpha(\alpha+1)}{2} + 2\frac{\alpha.\alpha}{2}$
   $= \frac{\alpha(3\alpha+1)}{2}$
   We want : $\frac{\alpha(3\alpha+1)}{2} = 2\alpha + 1$
   $3\alpha^2 + \alpha = 4\alpha + 2$
   $3\alpha^2 - 3\alpha - 2 = 0$
   Calculate $\Delta = 3^2 + 4 \cdot 3 \cdot 2 = 33$
   So : $\alpha = \frac{3\pm\sqrt{33}}{6}$. There is no entire solution : n is not f-perfect.

- If $n = p^\alpha$
  We want $\tau(n) - 1 = \tau(p) - 1 + \tau(p^2) - 1 + \cdots + \tau(p^{\alpha-1}) - 1$
  So $\alpha = 0 + 1 + \cdots + \alpha - 1$
  Hence $\alpha = \frac{\alpha(\alpha-1)}{2}$ which gives $1 = \frac{\alpha-1}{2}$ then $\alpha = 3$.

So an integer n is f-perfect for $f(n) = \tau(n) - 1)$ if and only if n is the cube of a prime number.

1.2.2 If $k = 1$

We want:

$$\tau(n) + 1 = \sum_{\substack{d|n \\ 1\leq d<n}} \tau(d) + 1$$

$$\tau(n) + 1 = \sum_{\substack{d|n \\ 1\leq d<n}} \tau(d) + \tau(n) - 1$$

$$2 = \sum_{\substack{d|n \\ 1\leq d<n}} \tau(d)$$

Study two cases :

- If n is a prime. We have:

$$\sum_{\substack{d|n \\ 1\leq d<n}} \tau(d) = \tau(1) = 1$$

  So n is not f-perfect.

- If n is at least the product of two prime numbers. We have:

$$\sum_{\substack{d|n\\1\le d<n}} \tau(d) \geq \tau(1)+\tau(p)+\tau(q) = 5$$

  So n is not f-perfect for $f(n)=\tau(n)+1$.

1.2.3 If $k \geq 2$

We want:

$$\begin{aligned}
\tau(n)+k &= \sum_{\substack{d|n\\1\le d<n}} \tau(d)+k\\
\tau(n)+k &= \sum_{\substack{d|n\\1\le d<n}} \tau(d)+k(\tau(n)-1)\\
(1-k)\tau(n)+2k &= \sum_{\substack{d|n\\1\le d<n}} \tau(d) > 0
\end{aligned}$$

So we obtain:

$$\begin{aligned}
2k &> (k-1)\tau(n)\\
2k &> k\tau(n)-\tau(n)\\
k(\tau(n)-2) &> -\tau(n)\\
k(\tau(n)-2) &< \tau(n)
\end{aligned}$$

Study two cases

- If $\tau(n)>2$ that is to say n not a prime.

  We obtain :

$$k < \frac{\tau(n)}{\tau(n)-2}$$

  Moreover if $\tau(n)>2$ then $\frac{\tau(n)}{\tau(n)-2}<2$. So there is a contradictory because $k\geq 2$ and we want $k<\frac{\tau(n)}{\tau(n)-2}$.

- If n is a prime.

  We obtain $\tau(n)=2$ and $f(n)=2+k$. Calculate:

$$\sum_{\substack{d|n\\1\le d<n}} \tau(d)+k = \tau(1)+k = 1+k$$

  So n is not f-perfect.

Consequently n is not f-perfect for $f(n)=\tau(n)+k$ if $k\geq 1$.

2 $f(n) = \varphi(n)$

Find all the f-perfect numbers n where $f(n) = \varphi(n)$ is the Euler's totient function.

2.1 Preliminaries

- Let p be a prime.
  Calculate $\varphi(p) = p^k - p^{(k-1)} = p^k(1 - \frac{1}{p})$

- Euler is a multiplicative function. Let $n = \prod_i p_i^{\alpha_i}$

  Prove Euler's product formula :

  Calculate :

$$\varphi(n) = \prod_i \varphi(p_i^{\alpha_i})$$

$$= \prod_i p_i^{\alpha_i}(1 - \frac{1}{p_i})$$

$$= \prod_i p_i^{\alpha_i} \prod_i (1 - \frac{1}{p_i})$$

$$\varphi(n) = n \prod_i (1 - \frac{1}{p_i})$$

- Euler's classical formula is :
$$\sum_{d|n} \varphi(d) = n$$

- Let p be a prime.
  $\varphi(n) = \prod_i (p_i - 1)p_i^{\alpha_i - 1}$
  Yet, if $p \neq 2, (p_i - 1) \equiv 0[2]$, and if $p = 2$, for all $\alpha > 1, p^{\alpha-1} \equiv 0[2]$
  So :
$$\forall n \geq 3, \varphi(n) \equiv 0[2]$$

  If n is φ-perfect, then $n \equiv 0[2]$ and $2|n$.

Solve, with Euler's classical formula :

$$n = \sum_{d|n} \varphi(d) = 2\varphi(n)$$

$$n = 2n \prod_i (1 - \frac{1}{p_i})$$

$$\prod_i (1 - \frac{1}{p_i}) = \frac{1}{2}$$

Yet $2|n$. So :

$$\prod_i (1 - \frac{1}{p_i}) \leq \frac{1}{2}$$

The only one possibility is $n = 2^\alpha$

n is φ-perfect if and only if $n = 2^\alpha$.

3 $f(n) = an + b$

3.1 Prove that if k is an integer such as $(2^{k+1} - 2k - 1)$ is a prime then $2^k(2^{k+1} - 2k - 1)$ is f-perfect for $f(n) = n - 1$

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Few notations:

$$P = 2^{k+1} - 2k - 1$$
$$N = 2^k P$$
$$S = \sum_{\substack{d|n \\ 1 \le d < n}} f(d)$$

We know that P is a prime. Consequently $D(n) = \{1, 2, \dots, 2^k, P, 2P, \dots, 2^k P\}$ and $\tau(n) = 2(k+1)$. Show that:

$$N - 1 = \sum_{\substack{d|N \\ 1 \le d < N}} d - 1$$

Calculate:

$$\begin{aligned}
\sum_{\substack{d|N \\ 1 \le d < N}} d - 1 &= \sum_{\substack{d|N \\ 1 \le d < N}} d - (\tau(N) - 1) \\
&= (1 + 2 + \dots + 2^k) + (P + 2P + \dots + 2^{k-1}P) - (2k + 1) \\
&= 2^{k+1} - 1 + P(2^k - 1) - 2k - 1 \\
&= 2^{k+1} - 2k - 1 + 2^k P - P - 1 \\
&= P + 2^k P - P - 1 \\
&= N - 1 \\
&= f(N)
\end{aligned}$$

So if P a prime we have effectively N f-perfect for $f(n) = n - 1$.

3.2 Find other similar sufficient conditions of f-perfectness for other polynomial functions of degree 1

3.2.1 For $f(n) = n + a$

Try with some values of k:

1. If $a = -2$

   Take $N = 2^k P$. Try to find conditions on P in order that N is f-perfect. Suppose that P is a prime.

Try to solve:

$$N-2=\sum_{\substack{d|N\\1\le d<N}} d-2$$

Simplify:

$$\begin{aligned}\sum_{\substack{d|N\\1\le d<N}} d-2 &= \sum_{\substack{d|N\\1\le d<N}} d-2(2k+1)\\ &= 2^{k+1}-1+(2^k-1)P-2(2k+1)\\ &= 2^{k+1}-1+2^kP-P-4k-2\\ &= 2^{k+1}-4k-1-P+2^kP-2\\ &= 2^{k+1}-4k-1-P+N-2\\ &= 2^{k+1}-4k-1-P+f(N)\end{aligned}$$

To have N f-perfect we need $2^{k+1}-4k-1-P=0$

So $P=2^{k+1}-4k-1$

So if $N=2^kP$ with P a prime and such as $P=2^{k+1}-4k-1$ then N is f-perfect for $f(n)=n-2$.

2. If $a=1$

Take $N=2^kP$. Try to find conditions on P again in order that N is f-perfect. Suppose that P is prime.

Try to solve:

$$N+1=\sum_{\substack{d|N\\1\le d<N}} d+1$$

Simplify:

$$\begin{aligned}\sum_{\substack{d|N\\1\le d<N}} d+1 &= \sum_{\substack{d|N\\1\le d<N}} d+2k+1\\ &= 2^{k+1}-1+(2^k-1)P+2k+1\\ &= 2^{k+1}-1+2^kP-P+2k+1\\ &= 2^{k+1}+2k-1-P+2^kP+1\\ &= 2^{k+1}+2k-1-P+N+1\\ &= 2^{k+1}+2k-1-P+f(N)\end{aligned}$$

To have N f-perfect we need $2^{k+1}+2k-1-P=0$ So $P=2^{k+1}+2k-1$

So if $N=2^kP$ with P a prime and such as $P=2^{k+1}+2k-1$ then N is f-perfect for $f(n)=n+1$.

3. Generalization

   Take $N = 2^k P$. Try to find conditions on P in order that N is f-perfect. Suppose that P is a prime.

   Try to solve:

$$N + a = \sum_{\substack{d|N \\ 1 \leq d < N}} d + a$$

   Simplify:

$$\begin{aligned}\sum_{\substack{d|N \\ 1 \leq d < N}} d + a &= \sum_{\substack{d|N \\ 1 \leq d < N}} d + (2k+1)a \\ &= 2^{k+1} - 1 + (2^k - 1)P + (2k+1)a \\ &= 2^{k+1} + 2ak - 1 - P + 2^k P + a \\ &= 2^{k+1} + 2ak - 1 - P + f(N)\end{aligned}$$

   To have N f-perfect we need $2k + 1 + 2ak - 1 - P = 0$ So $P = 2^{k+1} + 2ak - 1$

   So if $N = 2^k P$ with P a prime and such as $P = 2^{k+1} + 2ak - 1$ then N is f-perfect for $f(n) = n + a$.

3.2.2 If $f(n) = an$

Note $g(n) = n$ the identity function. So we have $f(n) = a \times g(n)$

Consequently, according to the preliminaries, the integers n which are g-perfect are also f-perfect.

3.2.3 If $f(n) = a(n + b)$

Note $g(n) = n + b$ So we have $f(n) = a \times g(n)$ Yet, according to the preliminaries ant the previous question, we have N g-perfect if $N = 2^k P$ with P a prime and such as $P = 2^{k+1} + 2bk - 1$.

So N is also f-perfect if $N = 2^k P$ with $P = 2^{k+1} + 2bk - 1$ and a prime.

4 $f(n) = \ln(n)$

Find all the ln-perfect integers.

We look for:

$$\ln(n) = \sum_{\substack{d|n \\ 1\leq d<n}} \ln(d) = \ln \prod_{\substack{d|n \\ 1\leq d<n}} d$$

The function ln is injective so:

$$n = \prod_{\substack{d|n \\ 1\leq d<n}} d$$

Study two different cases.

4.1 If $n = p^k$ with p a prime

We have $D(n) = \{1, p, p^2, \ldots, p^k\}$. So we obtain:

$$\prod_{\substack{d|n \\ 1\leq d<n}} d = \prod_{m=0}^{k-1} p^m = p^{\frac{k(k-1)}{2}}$$

We look for $k = \frac{k(k-1)}{2}$ that is to say $\frac{k-1}{2} = 1$ which gives to us a single solution $k = 3$. So we have:

- If $k = 3$ n is ln-perfect.
- If $k < 3$ n is ln-deficient.
- If $k > 3$ n is ln-abundant.

4.2 If $n = kpq$ with p and q different prime numbers

- If $k = 1$

  We have $D(n) = \{1, p, q, n\}$. So we obtain:

  $$n = pq = \prod_{\substack{d|n \\ 1\leq d<n}} d$$

  So n is ln-perfect.

- If $k \neq 1$

  We have $kp \mid n$ and $kq \mid n$. Then:

  $$n < kpkq < \prod_{\substack{d|n \\ 1\leq d<n}} d$$

  So n is ln-abundant.

Consequently the ln-perfect integers are the product of two different primes and the cubes of primes.

5 $f(n) = \omega^n$

Find all the f-perfect integers with $\omega = -1$.

We want:

$$f(n) = \sum_{\substack{d|n \\ 1 \leq d < n}} f(d)$$

We have $f(n) = -1$ if n is odd or $f(n) = 1$ if n is even.

Study different cases:

5.1 If n is odd.

We have $f(n) = -1$ and all the divisors of n are odd. So

$$\sum_{\substack{d|n \\ 1 \leq d < n}} f(d) = -(\tau(n) - 1)$$

We need $\tau(n) - 1 = 1$ that is to say $\tau(n) = 2$. So n has to be a prime.

5.2 If n is even.

We have $f(n) = 1$. We note $n = 2^k a$ with a odd. For every divisor of n odd we have k even divisors of n. So we have $\frac{\tau(n)}{k+1}$ odd divisors of n and $\frac{k(\tau(n))}{k+1} - 1$ even divisors (we put away n itself).

We obtain:

$$\sum_{\substack{d|n \\ 1 \leq d < n}} f(d) = -\frac{\tau(n)}{k+1} + \frac{k(\tau(n))}{k+1} - 1$$

Solve:

$$-\frac{\tau(n)}{k+1} + \frac{k(\tau(n))}{k+1} - 1 = 1$$

$$2\frac{k+1}{k-1} = \tau(n)$$

With k and $\tau(n)$ integers.

The only k which checked this equation are:

- $k = 5$ We obtain $\tau(n) = 3$ which is impossible because $k = 5 \implies \tau(n) \geq 6$
- $k = 3$ We obtain $\tau(n) = 4$ which is possible with $n = 2^3 = 8$
- $k = 2$ We obtain $\tau(n) = 6$ which is possible with $n = 2^2 a$ with a a odd prime.

Consequently n is f-perfect if and only if n is odd prime or n is such as $4p$ with p a prime.

6 $f(n) = \binom{m}{n}$

Let $f(n) = \binom{m}{n}$. Find all the f-perfect integers and study the general case where 2012 is replaced by a integer m.

Note $f_m(n) = \binom{m}{n}$.

If $m-1$ is prime, we have:

$$f_m(m-1) = \binom{m}{m-1} = m$$

And:

$$\sum_{\substack{d|m-1 \\ 1\leq d<m-1}} \binom{m}{d} = \binom{m}{1} = m$$

So we have:

- If $m-1$ is prime then $m-1$ is f-perfect for $f(n) = \binom{m}{n}$. So for the case $f_{2012}(n) = \binom{2012}{n}$ we have 2011 a prime so it is f_{2012}-perfect. We conjecture that there is no other f-perfect number.
- If $m-1$ is not a prime then we conjecture that there is no f-perfect number.

7 Other functions

Consider other arithmetic functions f and find necessary and/or sufficient conditions in order that an integer is f-perfect.

7.1 Let $f(n) = n^a$ with $a \geq 2$

Calculate:

$$\begin{aligned}\sum_{\substack{d|n\\1\leq d\leq n}} d^a &= \sum_{\substack{d|n\\1\leq d\leq n}} \left(\frac{n}{d}\right)^a \\ &= n^a \sum_{\substack{d|n\\1\leq d\leq n}} \frac{1}{d^a} \\ &\leq n^a \sum_{k=1}^{n} \frac{a}{k^a} \\ &< n^a \frac{\pi^2}{6} \\ &< 2\times n^a\end{aligned}$$

So there is no f-perfect integer.

7.2 Let $f(n) = a^n$ with $a \geq 2$

Calculate:

$$\sum_{\substack{d|n\\1\leq d<n}} a^d \leq \sum_{k=1}^{n-1} a^k = \frac{a^n-1}{a-1} \leq a^n - 1 < 2\times a^n$$

We can observe that if a is an integer and reasoning in base a, we have $f(a) = 10\cdots0$(with n zeros). And for all k $f(k) = 10\cdots0$ (with k zeros).
So :

$$\sum_{\substack{d|n\\1\leq d<n}} f(d) \leq \sum_{i<n} f(i) = \underbrace{1\cdots1}_{\text{with } n-1 \text{ one}} < 1\underbrace{0\cdots0}_{\text{with } n \text{ zeros}} = f(n)$$

So there is no f-perfect integer.

7.3 Let $f(n) = \mu(n)$ with μ the Moebius function

We call back that :

$$(n) = \begin{cases} 0 & \text{If } n \text{ a multiple of a square} \\ -1 & \text{If } n \text{ the product of an odd number of prime number} \\ 1 & \text{If } n \text{ the product of an even number of prime number}\end{cases}$$

Study two cases :
We call back that every divisor with one $\alpha_i > 1$ has for value 0 by μ.

- If n is such as :

$$n = \prod_{i=1}^{2k} p_i^{\alpha_i}$$

So we only keep the divisors which are such as :

$$d = \prod_{i=1}^{2k} p_i^{\gamma_i} \text{ with } \gamma_i \in \{0;1\}$$

The number of divisors having an even number of prime factors is :

$$\binom{2k}{0} + \binom{2k}{2} + \cdots + \binom{2k}{2k} = \frac{1}{2}\sum_{p=0}^{2k}\binom{2k}{p}1^p 1^{2k-p} = \frac{1}{2}(1+1)^{2k} = 2^{2k-1}$$

The number of divisors having an odd number of prime factors is :

$$\binom{2k}{1} + \binom{2k}{3} + \cdots + \binom{2k}{2k-1} = \frac{1}{2}\sum_{p=0}^{2k}\binom{2k}{p}1^p 1^{2k-p} = \frac{1}{2}(1+1)^{2k} = 2^{2k-1}$$

So :

$$\sum_{d|n} \mu(d) = 0$$

- If n is such as :

$$n = \prod_{i=1}^{2k+1} p_i^{\alpha_i}$$

So we only keep the divisors which are such as :

$$d = \prod_{i=1}^{2k+1} p_i^{\gamma_i} \text{ with } \gamma_i \in \{0;1\}$$

The number of divisors having an even number of prime factors is :

$$\binom{2k+1}{0} + \binom{2k+1}{2} + \cdots + \binom{2k+1}{2k} = \frac{1}{2}\sum_{p=0}^{2k+1}\binom{2k+1}{p}1^p 1^{2k+1-p} = \frac{1}{2}(1+1)^{2k+1} = 2^{2k}$$

The number of divisors having an odd number of prime factors is :

$$\binom{2k+1}{1} + \binom{2k+1}{3} + \cdots + \binom{2k+1}{2k-1} = \frac{1}{2}\sum_{p=0}^{2k+1}\binom{2k+1}{p}1^p 1^{2k+1-p} = \frac{1}{2}(1+1)^{2k+1} = 2^{2k}$$

So :

$$\sum_{d|n} \mu(d) = 0$$

Consequently we have for all $n \in \mathbb{N}^*$:

$$\sum_{d|n} \mu(d) = 0$$

The study of this function becomes largely more easier and we instantly have :

$$\mu(n) = 0 \implies \sum_{\substack{d|n \\ 1\leq d<n}} \mu(d) = 0 \implies \text{n } \mu\text{-perfect}$$

$$\mu(n) = -1 \implies \sum_{\substack{d|n \\ 1\leq d<n}} \mu(d) = 1 \implies \text{n } \mu\text{-abundant}$$

$$\mu(n) = 1 \implies \sum_{\substack{d|n \\ 1\leq d<n}} \mu(d) = -1 \implies \text{n } \mu\text{-deficient}$$

7.4 Let $f(n) = \frac{1}{n}$

We have $n \geq 2$. Consequently:

$$f(n) \leq \frac{1}{2}$$

Moreover 1 is one of the proper divisors of n and for every $d \mid n$ $f(d)$ s positive. So:

$$\sum_{\substack{d|n \\ 1\leq d<n}} \frac{1}{d} \geq 1$$

Finally we have :

$$f(n) < \sum_{\substack{d|n \\ 1\leq d<n}} f(d)$$

So there is no f-perfect integer.

8 Amicable couple

We call back that a couple $(m;n)$ of integers is said f-amicable if and if :

$$f(n) = \sum_{\substack{d|m \\ 1\leq d<m}} f(d)$$

and

$$f(m) = \sum_{\substack{d|n \\ 1\leq d<n}} f(d)$$

8.1 Preliminaries

- After the definition of a f-amicable couple we have :

  $$f(n) = \sum_{\substack{d|m \\ 1\leq d<m}} f(d) \Longleftrightarrow f(m) + f(n) = \sum_{d|m} f(d)$$

  We also have :

  $$f(m) = \sum_{\substack{d|n \\ 1\leq d<n}} f(d) \Longleftrightarrow f(n) + f(m) = \sum_{d|n} f(d)$$

  By combinaison we obtain that if the couple $(m;n)$ is f-amicable then :

  $$\sum_{d|n} f(d) = \sum_{d|m} f(d)$$

- Show that if $(a;b)$ is f and g-amicable then he is h-amicable for $h(n) = \alpha \times f(n) + \beta g(n)$ with $(\alpha;\beta) \in \mathbb{R}^2$.

  We have :

  $$f(a) = \sum_{\substack{d|b \\ 1\leq d<b}} f(d)$$
  $$f(b) = \sum_{\substack{d|a \\ 1\leq d<a}} f(d)$$
  $$g(a) = \sum_{\substack{d|b \\ 1\leq d<b}} d$$
  $$g(b) = \sum_{\substack{d|a \\ 1\leq d<a}} g(d)$$

Calculate first :

$$\begin{aligned} h(a) &= \alpha \times f(a) + \beta \times g(a) \\ &= \alpha \sum_{\substack{d|b \\ 1\le d<b}} f(d) + \beta \sum_{\substack{d|b \\ 1\le d<b}} g(d) \\ &= \sum_{\substack{d|b \\ 1\le d<b}} \alpha \times f(d) + \sum_{\substack{d|b \\ 1\le d<b}} \beta \times g(d) \\ &= \sum_{\substack{d|b \\ 1\le d<b}} \alpha \times g(d) + \beta \times f(d) \\ &= \sum_{\substack{d|b \\ 1\le d<b}} h(d) \end{aligned}$$

So we effectively have $(a;b)$ which is h-amicable for any combinaison of g and f.

- Let the next 3 properties :
  - P_1: $f(m) = f(n)$
  - P_2: m and n are f-perfect
  - P_3: $(m;n)$ is a f-amicable couple

We show that if two properties are truth then the third is also checked by the relation :

$$f(m) = \sum_{\substack{d'|n \\ 1\le d'<n}} f(d') = f(n) = \sum_{\substack{d|m \\ 1\le d<m}} f(d)$$

8.2 Study now different arithmetic functions

8.2.1 Let $f(n) = k \neq 0$

We want :

$$f(n) = \sum_{\substack{d|m \\ 1 \leq d < m}} f(d)$$

$$k = (\tau(n) - 1) \times k$$

With the same idea we have :

$$k = (\tau(m) - 1) \times k$$

Hence $\tau(m) = \tau(n) = 2$

So the f-amicable couples are such as $(p_1; p_2) \in \mathbb{P}^2$.

8.2.2 Let $f(n) = \tau(n)$

After the first question we know that the f-perfect integers are such as p^2 with p a prime. Moreover, the f-deficient integers are the primes. To have a f-amicable couple we need that one of the two integers be f-perfect or f-deficient. Study two cases

- Suppose m f-deficient so a prime. We have :

  $$\sum_{d|m} \tau(d) = \tau(1) + \tau(m) = 3$$

  Hence

  $$\sum_{d'|n} \tau(d') = 3$$

  We obtain that n is a prime. So $(m; n)$ will be a τ-amicable couple of f-deficient integers : it is impossible.

- Suppose m f-perfect an such as $m = p^2$ with p a prime. We have for all p belonging to P $\tau(p^2) = 3$.

So each couple such as $(p_1^2; p_2^2)$ with p_1 and p_2 primes is f-perfect.

8.2.3 $f(n) = \ln(n)$

We want :

$$\sum_{d|m} \ln(d) = \sum_{d'|n} \ln(d') = \ln(\prod_{d|m} d)$$
$$= \ln(\prod_{d'|n} d') = \prod_{d|m} d$$
$$= \prod_{d'|n} d'$$

Note :

$$P = \prod_{d|m} d$$

Calculate :

$$P^2 = \prod_{d|m} d \times \prod_{d|m} \frac{m}{d} = \prod_{d|m} m = m^{\tau(m)}$$

Hence $P = \sqrt{m^{\tau(m)}}$ We want consequently $\sqrt{m^{\tau(m)}} = \sqrt{n^{\tau(n)}}$. m and n have the same prime factors in their decomposition. Note $m = p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k}$ and $n = p_1^{\beta_1} \times \cdots \times p_k^{\beta_k}$ We look for : $\forall i, \alpha_i \times \tau(m) = \beta_i \times \tau(n)$

Suppose that there is one i where $\alpha_i > \beta_i$. We obtain that $\tau(m) < \tau(n)$ hence $\forall i, \alpha_i > \beta_i$, so $\tau(m) > \tau(n)$.

So by the absurde we have a contradiction. So $m = n$.

Consequently there is no ln-amicable couple of integers.

8.2.4 Let $f(n) = -1^n$

We have for all k $f(k)1$ Study two different cases :

- If m is odd we have $f(m) = -1$ and all its divisors are odd too. m is f-deficient and :

$$f(n) = \sum_{\substack{d|m \\ 1 \leq d < m}} f(d) = -\tau(m) + 1$$

  The only solution is consequently $\tau(m) = 2$ So m is f-perfect. So n has to be also f-perfect. So we have the couple $(m;n) \in \mathbb{P}^2$ which is solution if n and m are different of 2.

- If m is even we have :

$$f(m) = 1 = \sum_{\substack{d'|n \\ 1 \leq d' < n}} f(d')$$

  So n has a even divisor and is even itself. Then we have $f(n) = 1 = f(m)$. In order to have $(m;n)$ amicable we need n and m f-perfect so such as $4p$ with p a prime.

So the f-amicable couple are such as :

- $(p_1; p_2)$ with p_1 and p_2 odd primes.
- $(4p_1; 4p_2)$ with p_1 and p_2 primes.

8.2.5 Let the functions $f(n) = \frac{1}{n}$ $f(n) = a^n$ $f(n) = n^a$ with $a \geq 2$

As seen before all the integers are f-abundant for those functions.

So there is no f-amicable couple.

Conclusion : We constate that for many of the functions the f-amicable couples are couples of f-perfect integers which checked $f(m) = f(n)$.