6. Recurrent sequences

ITYM Team Croatia

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Abstract

We have completely analysed the first task, and have proved all parts of it. In the second task, we have proved very little, but have some strong theories that we have created from the trial and error with a coded generator of the sequence. We have, if our theories of the second task are correct, more or less completely solved the a) part of the third task, and have also solved the b) part of the task.

1 The sequence of squares

We have found the conditions for which the sequence $u_{n+1} = \frac{u_1^2+u_2^2+...+u_n^2}{n}$ is convergent, for which it is divergent, and we have proved that it is always monotonic. First, let us write the sequence in a more convenient form:

$$u_{n+1} = \frac{(n-1)u_n + u_n^2}{n}.$$

First case: $u_3 < 1$ In case $u_3 < 1$ the sequence is monotonically decreasing and converges into 0.

Proof of monotonicity Let's assume that we have already proved that $u_i > u_{i+1}, \forall i$ such that $3 \leq i \leq n$. To complete the induction we need to prove that $u_{n+1} < u_n$.

We have

$$\begin{aligned}
\frac{u_1^2 + ... + u_n^2}{n} &< \frac{u_1^2 + ... + u_{n-1}^2}{n-1}, \\
(n-1)(u_1^2 + ... + u_n^2) &< n(u_1^2 + ... + u_{n-1}^2), \\
(n-1)u_n^2 &< u_1^2 + ... + u_{n-1}^2, \\
u_n^2 &< \frac{u_1^2 + ... + u_{n-1}^2}{n-1}, \\
u_n^2 &< u_n,
\end{aligned}$$

which holds because $u_3 < 1$.

Since the sequence is bounded and monotonic, it is obviously convergent, so we have found its limit:

$$\begin{aligned} nL &= (n-1)L + L^2 \\ &= nL - L + L^2, \end{aligned}$$

and we have $L = L^2$, so the solution we are looking for is obviously 0.

Second case: $u_3 > 1$ In case $u_3 > 1$ the sequence is monotonically increasing and is divergent.

Proof of monotonicity Let's assume that we have allready proved that $u_i < u_{i+1} \forall i$ such that $3 \leq i \leq n-1$. To complete the induction we need to prove that $u_{n+1} > u_n$.

We have

$$\begin{aligned} \frac{u_1^2 + ... + u_n^2}{n} &> \frac{u_1^2 + ... + u_{n-1}^2}{n-1}, \\ (n-1)(u_1^2 + ... + u_n^2) &> n(u_1^2 + ... + u_{n-1}^2), \\ (n-1)u_n^2 &> u_1^2 + ... + u_{n-1}^2, \\ u_n^2 &> \frac{u_1^2 + ... + u_{n-1}^2}{n-1}, \\ u_n^2 &> u_n, \end{aligned}$$

which holds because $u_3 > 1$.

Proof of divergence We will prove that the difference between every two succeeding elements is increasing. In order to make the proof easier to read we will define

$$S_n = u_1^2 + u_2^2 + ... + u_n^2.$$

Now we have the following:

$$\begin{aligned} u_{n+2} - u_{n+1} &> u_{n+1} - u_n, \\ u_{n+2} + u_n &> 2u_{n+1}, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} n(n-1)S_{n+1} + n(n+1)S_{n-1} &> 2(n-1)(n+1)S_n, \\ n^2u_{n+1}^2 - nu_n^2 - nu_{n+1}^2 &> n^2u_n^2 - 2S_n, \\ n^2u_{n+1}^2 - nu_n^2 - nu_{n+1}^2 &> n2u_n^2 - 2nu_{n+1}. \end{aligned}$$

By solving this quadratic equation for u_{n+1}, we get the discriminant

$$\begin{aligned} D &= 4 - 4(n-1)(n+1), \\ &= 8 - 4n^2, \end{aligned}$$

which is negative for all $n > 1$.

This means the inequality holds, and that the sequence is divergent.

Third case: $u_3 = 1$ In case $u_3 = 1$ every term of the sequence equals one.

Proof Let us assume that we have already proved this for the first n elements. Then we have $u_{n+1} = \frac{u_1+...+u_n}{n} \Longrightarrow u_{n+1} = \frac{n}{n} = 1$.

2 The rotated sequence

For the second sequence we haven't been able to deternime all of these properties, but we have noticed that it would be of most use to determine for which positive reals this sequence diverges. This would be useful becuase this is the minimal sequence that we can get from numbers $u_1, u_2, ..., u_n$, and the sequence from the first task is the maximal, as we will later prove using induction and the rearrangement inequality. This would be of great help in solving the a part of the third problem. However, we haven't been able to determine whether this holds; the only thing we have proved for this sequence is the following:

Claim Define two sequences (u_n) and (v_n) in the given way, which sastisfy the following:

- $u_1 = x, u_2 = y,$
- $v_1 = -x, v_2 = y,$

Then the equality $u_n = (-1)^n v_n$ holds.

Proof by induction: Let us assume we have allready proved this for the first n elements of the 2 sequences, then $u_{n+1} = \frac{u_1u_n+u_2u_{n-1}+...+u_nu_1}{n}$ and
$v_{n+1} = \frac{v_1v_n+...+v_nv_1}{n}$
$v_{n+1} = \frac{(-1)^1u_1(-1)^nu_n+(-1)^2u_2(-1)^{n-1}u_{n-1}+...+(-1)^nu_n(-1)^1u_1}{n}$
$v_{n+1} = (-1)^{n+1}\frac{u_1u_n+u_2u_{n-1}+...+u_nu_1}{n}$
$v_{n+1} = (-1)^{n+1}u_{n+1}$

3 The random sequence

Now let us prove that every sequence in the third problem is somewhere between the sequences defined in problems one and two. Let's call the sequence from the first task sequnce $(a_n)_n$ and the sequnce defined in the second task $(b_n)_n$.

Remark From this point on, we will only be working with positive real numbers. We will also assume that $u_1 = a_1 = b_1$ and $u_2 = a_2 = b_2$.

Claim We are going to prove that $a_n \geq u_n \geq b_n$.

Since $a_3 \geq b_3$ is equivalent to $a_1^2 + a_2^2 \geq 2a_1a_2$, it is easy to prove that it is true. This will be the base of our induction.

Proof that $a_n \geq u_n$ (induction) Let us assume that we have proved that $a_i \geq u_i \forall i 3 \leq i \leq n$.
Then it is obvious that $\frac{u_1 u_{f(1)} + u_2 u_{f(2)} + ... + u_n u_{f(n)}}{n} \leq \frac{a_1 a_{f(1)} + a_1 a_{f(2)} + ... + a_n a_{f(n)}}{n}$
And then by applying the rearrangement inequality we get $a_1 a_{f(1)} + a_2 a_{f(2)} + ... + a_n a_{f(n)} \leq a_1^2 + a_2^2 + ... + a_n^2$ and so
$\frac{a_1 a_{f(1)} + a_2 a_{f(2)} + ... + a_n a_{f(n)}}{n} \leq \frac{a_1^2 + a_2^2 + ... + a_n^2}{n}$
This completes the step of the induction so we get $a_{n+1} \geq u_{n+1}$

Proof that $a_n \leq u_n$ (induction) Let us assume that we have proved that $b_i \leq u_i \forall i 3 \leq i \leq n$.
Then it is obvious that

$$\frac{u_1 u_{f(1)} + u_2 u_{f(2)} + ... + u_n u_{f(n)}}{n} \geq \frac{b_1 b_{f(1)} + b_1 b_{f(2)} + ... + b_n b_{f(n)}}{n}$$

And then by applying the rearrangement inequality we get

$$\begin{aligned} b_1 b_{f(1)} + b_2 b_{f(2)} + ... + b_n b_{f(n)} &\geq b_1 b_n + b_2 b_{n-1} + ... + b_n b_1, \\ \frac{b_1 b_{f(1)} + b_2 b_{f(2)} + ... + b_n b_{f(n)}}{n} &\geq \frac{b_1 b_n + b_2 b_{n-1} + ... + b_n b_1}{n}. \end{aligned}$$

This completes the step of the induction so we get

$$b_{n+1} \leq u_{n+1}.$$

From this we can get several necessary and sufficient conditions for divergence and convergence.

Corollary 1 The necessary condition for the sequence u_n to be divergent is the same as the condition for the sequence a_n to be divergent, $u_1^2 + u_2^2 > 2$.

Corollary 2 The sufficient condition for the sequence u_n to be convergent is the same as the condition for the sequence a_n to be convergent, $u_1^2 + u_2^2 < 2$.

We have reasons to believe that the following holds:

Hypothesis 1 The sufficient condition for the sequence u_n to be divergent is the same as the condition for the sequence b_n to be divergent.

Hypothesis 2 The necessary condition for the sequence u_n to be convergent is the same as the condition for the sequence b_n to be convergent.

4 The random function

We have also found several conditions for the sequence defined with the random function to be divergent and convergent.

Proposition 1 If $u_1 + u_2 > 2$ the sequence can be divergent for some values of the function.

Proof Let us first prove that if $u_1 + u_2 > 2$ the sequence can be divergent. Obviously, since we are still dealing only with positive reals, the easiest way for this function to be divergent is for it to attain its maximal value each time.
This way, if we mark the greates element of the first n in the sequence as u_{max}, we can make $u_{n+1} = \frac{u_1 u_{f(1)} + u_2 u_{f(2)} + ... + u_n u_{f(n)}}{n}$ become $u_{n+1} = u_{max}\frac{u_1+u_2+...+u_n}{n}$, and since $u_1 + u_2 > 2$, we know that $u_3 > u_1$ and $u_3 > u_2$.
We will use this as the base of our induction. Now let us assume that we have already proved that $u_i > u_{i-1} \forall i 3 \leq i \leq n$
We have to prove that
$u_{n+1} > u_n$
$u_{max}\frac{u_1+u_2+...+u_n}{n} > u_{max}\frac{u_1+u_2+...+u_{n-1}}{n-1}$
$(n-1)(u_1 + u_2 + ... + u_n) > n(u_1 + u_2 + ... + u_{n-1})$
$(n-1)u_n > u_1 + u_2 + ... + u_{n-1}$
$u_n > \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{max}\frac{u_1+u_2+...+u_{n-1}}{n-1} > \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{max} > 1$ which we know is true.

Proposition 2 If $u_1 + u_2 < 2$ the sequence can converge for some values of the function.

Proof Now let su prove that the sequence can converge to 0 for $u_1+u_2 < 2$.

Obviously, the easiest way for this to happen is if the random function attains the minimal value every time.
Let us mark the minimal element of the first n elements of the sequence as u_{min}.
So, we can write the $n+1$st element of the sequence as $u_{n+1} = u_{min}\frac{u_1+u_2+...+u_n}{n}$.

$u_3 = u_{min}\frac{u_1+u_2}{2}$ and since $u_1 + u_2 < 2$ we know that $u_3 < u_{min}$, we will use this as the base of the induction.
Now let us assume that we have already proved that $u_i < u_{i-1} \forall i 3 \leq i \leq n$
We have to prove that
$u_{n+1} < u_n$
$u_{min}\frac{u_1+u_2+...+u_n}{n} < u_{min}\frac{u_1+u_2+...+u_{n-1}}{n-1}$
$(n-1)(u_1 + u_2 + ... + u_n) < n(u_1 + u_2 + ... + u_{n-1})$
$(n-1)u_n < u_1 + u_2 + ... + u_{n-1}$
$u_n < \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{min}\frac{u_1+u_2+...+u_{n-1}}{n-1} < \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{min} < 1$ which we know is true.

Proposition 3 If both u_1 and u_2 are greater than 1, the sequence will be divergent for all values of the function.

Proof Now let su prove that regardless of what values the function attains, the sequence will be divergent if both $u_1 > 1$ and $u_2 > 1$.
It is obvious that the sequence will will attain the least possible value if the function attains the least possible value each time.
Let us mark the smallest element of the first n elements as u_{min} ,this way we can make $u_{n+1} = \frac{u_1 u_{f(1)} + u_2 u_{f(2)} + ... + u_n u_{f(n)}}{n}$ become $u_{n+1} = u_{min} \frac{u_1 + u_2 + ... + u_n}{n}$.
Now let us assume that we have already proved that $u_i > u_{i-1} \forall i 3 \leq i \leq n$
We have to prove that
$u_{n+1} > u_n$
$u_{min} \frac{u_1+u_2+...+u_n}{n} > u_{min} \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$(n-1)(u_1 + u_2 + ... + u_n) > n(u_1 + u_2 + ... + u_{n-1})$
$(n-1)u_n > u_1 + u_2 + ... + u_{n-1}$
$u_n > \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{min} \frac{u_1+u_2+...+u_{n-1}}{n-1} > \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{min} > 1$ which we know is true.

Proposition 4 If both u_1 and u_2 are smaller that 1, the sequence will converge for all values of the function.

Proof Now let su prove that regardless of what values the function attains, the sequence will be convergent if both $u_1 < 1$ and $u_2 < 1$.
Obviously the sequence will attain the greatest possible value if the function attains the greatest possible value each time.
Let us mark the maximal element of the first n elements of the sequence as u_{max}.
So, we can write the $n+1$st element of the sequence as $u_{n+1} = u_{max} \frac{u_1+u_2+...+u_n}{n}$.

Now let us assume that we have already proved that $u_i > u_{i-1} \forall i 3 \leq i \leq n$
We have to prove that
$u_{n+1} < u_n$
$u_{max} \frac{u_1+u_2+...+u_n}{n} < u_{max} \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$(n-1)(u_1 + u_2 + ... + u_n) < n(u_1 + u_2 + ... + u_{n-1})$
$(n-1)u_n < u_1 + u_2 + ... + u_{n-1}$
$u_n < \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{max} \frac{u_1+u_2+...+u_{n-1}}{n-1} < \frac{u_1+u_2+...+u_{n-1}}{n-1}$
$u_{max} < 1$ which we know is true.