4. Isosceles Triangles

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Abstract

The key tool for solving most of problems was the Stewart's theorem: $b^2m + c^2n = a(mn + d^2)$, because it expresses the the length of specfic line in dependance of lengths of sides of triangle. When solving problem for internal angle bisector we used angle bisector theorem: $\frac{m}{n} = \frac{a}{b}$, and in the second problem we used the Steiner's theorem: $\frac{|BA_1||BA_2|}{|CA_1||CA_2|} = \frac{|AB|^2}{|AC|^2}$.

We proved the equivalence for internal angle bisector, symmedian and external angle bisector. Also, we stated basic concepts considering n-lines.

1 Angle bisectors and symmedians

Problem 1. Prove that two internal angle bisectors of a triangle are equal if and only if the triangle is isosceles.

Solution: Using Stewart's theorem, we have that $c(d_c^2 + pq) = a^2q + b^2p$ (1), where p and q are lenght of the segments of a side of length c closer to the a and b, respectively. Because $\frac{p}{q} = \frac{a}{b}$ and $p + q = c$, we have that $p = \frac{ac}{a+b}$ and $q = \frac{bc}{a+b}$. If we put this in (1), we get $d_c = \frac{\sqrt{ab((a+b)^2-c^2)}}{a+b}$ (that is, the lenght of an internal angle bisector through the vertice C is equal to $\frac{\sqrt{ab((a+b)^2-c^2)}}{a+b}$). If the triangle is isosceles, we apply this formula for angle bisectors from vertices B and C (we assume that $b = c$). It follows that $d_b = \frac{\sqrt{ac((a+c)^2-b^2)}}{a+c} = \frac{\sqrt{ab((a+b)^2-c^2)}}{a+b} = d_c$,because $b = c$. Now we have to prove that $d_b = d_c$implies $b = c$. We have:

$$\frac{\sqrt{ac((a+c)^2-b^2)}}{a+c} = \frac{\sqrt{ab((a+b)^2-c^2)}}{a+b}$$

$$\frac{c((a+c)-b^2)}{(a+c)^2} = \frac{b((a+b)^2-c^2)}{(a+b)^2}$$

$$(a+b)^2c((a+c)^2-b^2) = (a+c)^2b((a+b)^2-c^2)$$

$(a^2c + 2abc + b^2c)(a^2 + 2ac + c^2 - b^2) = (a^2b + 2abc + bc^2)(a^2 + 2ab + b^2 - c^2)$

$a^4c+2a^3c^2+a^2c^3-a^2b^2c+2a^3bc+4a^2bc^2+2abc^3-2ab^3c+a^2b^2c+2ab^2c^2+b^2c^3-b^4c =$

$= a^4b+2a^3b^2+a^2b^3-a^2bc^2+2a^3bc+4a^2b^2c+4ab^3c-4abc^3+a^2bc^2+2ab^2c^2+b^3-bc^4$

$(b-c)(a^4+2a^3(b+c)+a^2(b^2+bc+c^2)+4a^2bc+4abc(b+c)+b^2c^2+bc(b^2+bc+c^2)) = 0$

Because $(a^4+2a^3(b+c)+a^2(b^2+bc+c^2)+4a^2bc+4abc(b+c)+b^2c^2+bc(b^2+bc+c^2))$ is strictly positive, we conclude that $b = c$.

Problem 2. The symmedian through a given vertex of a triangle is constructed by reflecting the median about the internal angle bisector at the same vertex. Prove that two symmedians of a triangle are equal if and only if the triangle is isosceles.

Solution: We use Stewart's theorem and the fact that $\frac{p}{q} = \frac{c^2}{b^2}$, where p and q are lengths of the segments on the side a, where p is closer to the c, and q to b.

We get that $p = \frac{ac^2}{b^2+c^2}$ and $q = \frac{ab^2}{b^2+c^2}$. Because $a(l_a^2 + pq) = b^2p + c^2q$, where $l_a$is length of a symmedian through A, we have $l_a^2 = \frac{2b^2c^2}{b^2+c^2} - \frac{a^2b^2c^2}{(b^2+c^2)^2}$.

Similarly, $l_b^2 = \frac{2a^2c^2}{a^2+c^2} - \frac{a^2b^2c^2}{(a^2+c^2)^2}$. Now, if $a = b$, then $l_a^2 = \frac{2b^2c^2}{b^2+c^2} - \frac{a^2b^2c^2}{(b^2+c^2)^2} = \frac{2a^2c^2}{a^2+c^2} - \frac{a^2b^2c^2}{(a^2+c^2)^2} = l_b^2$, and we have $l_a = l_b$.

If $l_a = l_b$, then:

$$\frac{2b^2c^2}{b^2 + c^2} - \frac{a^2b^2c^2}{(b^2 + c^2)^2} = \frac{2a^2c^2}{a^2 + c^2} - \frac{a^2b^2c^2}{(a^2 + c^2)^2}$$

$$\frac{2b^2}{b^2 + c^2} - \frac{2a^2}{a^2 + c^2} = \frac{a^2b^2}{(b^2 + c^2)^2} - \frac{a^2b^2}{(a^2 + c^2)^2}$$

$$\frac{2b^2(a^2 + c^2) - 2a^2(b^2 + c^2)}{(b^2 + c^2)(a^2 + c^2)} = \frac{a^2b^2((a^2 + c^2)^2 - (b^2 + c^2)^2)}{(b^2 + c^2)^2(a^2 + c^2)^2}$$

$$2c^2(b^2 + c^2)(a^2 + c^2)(b^2 - a^2) = a^2b^2(a^2 - b^2)(a^2 + b^2 + 2c^2)$$

$$(b^2 - a^2)(2c^2(b^2 + c^2)(a^2 + c^2) + a^2b^2(a^2 + b^2 + 2c^2)) = 0$$

Because $(2c^2(b^2 + c^2)(a^2 + c^2) + a^2b^2(a^2 + b^2 + 2c^2))$ is strictly positive, we conclude that $b^2 = a^2$, and finally $b = a$.

Problem 3 Is it true that two external angle bisectors of a triangle are equal if and only if the triangle is isosceles?

Solution: Let CD be an external angle bisector (D is the point where external angle meets the line AB). d_c is the length of line CD. By the sine rule, we have $\frac{\sin(90+\frac{\gamma}{2})}{c+AD} = \frac{\sin\sphericalangle ADC}{a}$ and $\frac{\sin(90-\frac{\gamma}{2})}{AD} = \frac{\sin\sphericalangle ADC}{b}$. Because $\sin(90+\frac{\gamma}{2}) = \sin(90-\frac{\gamma}{2})$, we get $d_c = \frac{bc}{a-b}$. In the same way we get $d_a = \frac{ab}{c-b}$, where d_a is length of an external angle bisector through A. (Of course, with the assumptions that $a > b$ and $c > b$, other cases are treated similarly).

Using Stewart's theorem, we get: $BD(b^2 + \frac{bc^2}{a-b}) = \frac{a^2bc}{a-b} + cd_c^2$, where $BD = \frac{ac}{a-b}$, and from here: $d_c^2 = \frac{abc^2}{(a-b)^2} - ab$. Similarly we have: $d_a^2 = \frac{a^2bc}{(c-b)^2} - bc$. Now, if $a = c$, then we get $d_a = d_c$. If $d_a = d_c$, we have:

$$\frac{a^2bc}{(c-b)^2} - bc = \frac{abc^2}{(a-b)^2} - ab$$

$$\frac{a^2c}{(c-b)^2} - c = \frac{ac^2}{(a-b)^2} - a$$

$$a - c = ac\left(\frac{c}{(a-b)^2} - \frac{a}{(c-b)^2}\right)$$

$$c - a = \frac{-ac}{(a-b)^2(c-b)^2}\left((c-a)(c^2 + ac + a^2 - 2b(a+c) + b^2)\right)$$

Now we have two cases. In the first case $c = a$, (triangle is isosceles). Second case is when the triangle is not isosceles, so we can divide both sides with $c - a$. Then:

$$(a-b)^2(c-b)^2 = (2ab + 2bc - c^2 - ac - a^2 - b^2)ac$$

Because the left side can't be negative, so the right side must be negative, either.

$$2ab + 2bc \geqslant a^2 + b^2 + c^2 + ac$$

We use inequality $a^2 + b^2 \geqslant 2ab$ to get:

$$2ab + 2bc \geqslant a^2 + b^2 + c^2 + ac \geqslant 2ab + c^2 + ca$$

$$2bc \geqslant c^2 + ca$$

$$2b \geqslant a + c$$

But we assumed that $a > b$ and $c > b$ which implies $a + c > 2b$, so the second case is not possible, which means that $c = a$ (triangle is isosceles). Now we proved that two external angle bisectors of a triangle are equal if and only if the triangle is isosceles.

2 The $n-$lines

Problem 5 Check that the internal bisectors and the symmedians are respectively internal 1-lines and 2-lines of the triangle. Also, the external bisectors and the exsymmedians are respectively external 1-lines and 2-lines of the triangle.

Solution: Let m and n be lenght of the segments of a side of length a closer to the b and c, respectively, and d the length of internal bisector through the vertice A. By the sine rule, we have $\frac{sin\frac{\alpha}{2}}{n} = \frac{sin(180-\frac{\alpha}{2}-\gamma)}{c}$ and $\frac{sin\frac{\alpha}{2}}{m} = \frac{sin(\frac{\alpha}{2}+\gamma)}{b}$. As $sin(180 - \frac{\alpha}{2} - \gamma) = sin(\frac{\alpha}{2} + \gamma)$, $\frac{m}{n} = \frac{b}{c}$. We have proved that internal bisector isan internal 1-line od triangle.

Let the $\delta = \measuredangle A_2CA_1 = \measuredangle A_1CA_0$. By the sine rule$\frac{|AA_2|}{sin(\frac{\alpha}{2}-\delta)} = \frac{|AC|}{sin\measuredangle AA_2C}$ and $\frac{|BA_2|}{sin(\frac{\alpha}{2}+\delta)} = \frac{|BC|}{sin\measuredangle CA_2B}$. Also, $\frac{|AA_0|}{sin(\frac{\alpha}{2}+\delta)} = \frac{|AC|}{sin\measuredangle AA_1C}$ and $\frac{|BA_0|}{sin(\frac{\alpha}{2}-\delta)} = \frac{|AC|}{sin\measuredangle CA_1B}$. As $sin\measuredangle AA_2C = sin\measuredangle CA_2B$, we get $\frac{|AD|}{|BD|} = \frac{|AC|*sin(\frac{\alpha}{2}-\delta)}{|BC|*sin(\frac{\alpha}{2}+\delta)}$ (2). As $|AA_0| = |BA_0|$, we get $\frac{sin(\frac{\alpha}{2}-\delta)}{sin(\frac{\alpha}{2}+\delta)} = \frac{|AC|}{|BC|}$. When we put this in (2) we get $\frac{|AD|}{|BD|} = \frac{|AC|^2}{|BC|^2}$, and we proved that symedians are internal 2-lines of the triangle.

Problem 6 Is it true that two internal n-lines of triangle are equal if and onla if the triangle is isosceles?

Solution: Let d be the length of n-line and m and n be the length of segments BA_n and A_nC, respectively. Using Stewart's theorem, we have that $b^2m + c^2n = a(mn + d^2)$ (3). Because of definition of n-lines $\frac{m}{n} = \frac{c^k}{b^k}$ and, of course, $m + n = a$. From this we get $m = \frac{ac^k}{b^k+c^k}$ and $n = \frac{ab^k}{b^k+c^k}$. When we put this in (3) we get:

$$\frac{b^2c^k + b^kc^2}{b^k + c^k} = d^2 + \frac{a^2b^kc^k}{(b^k + c^k)^2}.$$