1. Generalizing perfectness

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Abstract

The main aspect of this problem is finding f-perfect numbers. Then we have a variety of functions. For almost each function we have examples that guide us to the solution. The most problematic functions were the root of unity and binome coefficients. Also, we managed to use Fibonacci's numbers and found f-perfect numbers for Fibonacci's numbers. We solved most of this task but some generalizations are missing.

Problem 1 $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ where$p_i = p_j \iff i = j$

$\alpha_i \geq \alpha_{i+1}$ $m = \frac{n}{p_1^{\alpha_1}}$

a) Problem: Prove that a natural number n $\geq$ 1 is τ -perfect if and only if n is the square of a

prime.

$\tau(n) = \tau(p_1^{\alpha_1}.m) = (\alpha_1{+}1).\tau(m) = \sum_{\binom{d|n}{1\leq d\leq n-1}} \tau(d) {\geq} \frac{\alpha_1.(\alpha_1+1).\tau(m)}{2} {\Longrightarrow} 2 {\geq}\ \alpha_1$

$1^{\circ}\alpha_1 = 1$

$n = \prod_{i=1}^{k} p_i^{\alpha_i}$

$\tau(n) = 2^k = \sum_{i=0}^{k-1} \binom{k}{i} \cdot 2^i$

$2^k + 2^k = \sum_{i=0}^{k} \binom{k}{i} \cdot 2^i$

$2^{k+1} = 3^k$ there are no solutions $\in$

$2^{\circ}\alpha_1 = 2$

$n = p_1^2 . m$

m and pm are divisors of $n \Longrightarrow \tau(n) = 3 \cdot \tau(m) \geq \tau(m) + 2 \cdot \tau(m) \cdot \tau(n)$ is equal to the sum of

τ of all its divisors and to the sum of $\tau(m)$ and $\tau(pm)$ergo m and pm are the only divisors of n

ergo m is 1 ergo $n = p^2$

$\tau(p^2) = \tau(p) + \tau(1)$

$3 = 2 + 1$

b) Problem: Find all f -perfect natural numbers n $\geq$ 1 for the function f (n) = τ (n) − 1.

For as many values of k $\in$ Z as possible, find all f -perfect natural numbers n $\geq$ 1 for

f (n) = τ (n)+k.

$\tau(n) - 1 = \sum_{\binom{d|n}{1\leq d\leq n-1}} \tau(d) - (\tau(n) - 1)$

$(\alpha_1+1) \centerdot \tau(m) - 1 \geq \frac{\alpha_1(\alpha_1+1)}{2}\tau(m) - \alpha_1$

$\alpha_1 - 1 \geq \frac{(\alpha_1-2)(\alpha_1+1)}{2}\tau(m) \Longrightarrow \alpha_1 - 1 \geq \frac{(\alpha_1-2)(\alpha_1+1)}{2} \Longrightarrow 3 \geq \alpha_1$

$1^\circ \alpha_1 = 1$

$\tau(n) = 2^k$

$2^k - 1 = \sum_{i=0}^{k-1} \binom{k}{i} \cdot 2^i - (2^k - 1)$

$2 \centerdot 2^k - 2 + 2^k = \sum_{i=0}^{k-1} \binom{k}{i} \cdot 2^i + 2^k$

$3 \centerdot 2^k - 2 = 3^k \Rightarrow n = 1$

$2^\circ \alpha_1 = 2$

$\tau(n) - 1 = 3 \centerdot \tau(m) - 1 = \tau(m) - 1 + 2 \centerdot \tau(m) - 1 + r \Rightarrow r = 1 \Rightarrow$n has exactly 1 more divisor

and it is a prime.

a) The divisor is $p \Rightarrow m \neq 1 \Rightarrow m$ must be a prime$\Rightarrow$then p^2 is also a divisor what is

impossible due to the primar hypothesis.

b) The divisor is a divisor of $m \Rightarrow m$ is the divisor$\Rightarrow$ p^2 is a divisor of n what is

impossible due to the primary hypothesis.

$3^\circ \alpha_1 = 2$

$\tau(n) - 1 = 4 \centerdot \tau(m) - 1 = \tau(m) - 1 + 2 \centerdot \tau(m) - 1 + 3 \centerdot \tau(m) - 1 + r$

$r = 2 - 2 \centerdot \tau(m)$

$m = 1, r = 0$

$n = p^3$

Problem 2 $f(n) = \varphi(n)$

We know that $\sum_{d|n} \varphi(d) = n$

$\varphi(n) = \sum_{\binom{d|n}{1 \leq d \leq n-1}} \varphi(d)$

$2 \centerdot \varphi(n) = \sum_{d|n} \varphi(d) = n$

$2 \centerdot \varphi(n) = n$

$\varphi(n) = \frac{n}{2}$

$n \centerdot \prod_{p|n}(1 - 1/p) = \frac{n}{2}$

$\prod_{p|n}(1 - 1/p) = \frac{1}{2}$

$p = 2 \Rightarrow n = 2^\alpha$

Also,for every $\alpha \in N, \varphi(2^\alpha) = 2^{\alpha-1}$.Therefore,

$\varphi(n) = \sum_{\binom{d|n}{1 \leq d \leq n-1}} \varphi(d)$

$2^{\alpha-1} = 1 + \sum_{i=0}^{\alpha-2} 2^\alpha = 2^{\alpha-1}$

Problem 3 a) Prove that if k is a natural number such that $2^{k+1} - 2k - 1$ is a prime, then

$n = 2^k \centerdot (2^{k+1} - 2k - 1)$ if f-perfect for $f(n) = n - 1$

g$(n) = \sum_{\binom{d|n}{1 \leq d \leq n-1}} f(d)$

$g(n) = (2^k - 1) \centerdot p - (2k+1) + 2^k - 1 = n - 1$

$p \centerdot 2^k - 1 = p \centerdot 2^k - p - 2k - 1 + 2^{k+1} - 1$

$p = 2^{k+1} - 2k - 1$

b) Find similar sufficient conditions for f -perfectness for other polynomial functions

of degree1 such as

$f(n) = n - 2$ or $f(n) = n + 1$

$f(n) = n + a$

$n = p.2^k$

$f(n) = g(n)$

$p.2^k + a = (2^k - 1).p + 2^{k+1} - 1 + a(2k + 1)$

$p = 2^{k+1} + 2ak - 1$

Problem 4 $f(n) = \text{lnn}$

$\ln(n) = \sum_{\left(\substack{d|n \\ 1 \leq d \leq n-1}\right)} \ln(d)$

$\ln n = \ln\left(\prod_{\left(\substack{d|n \\ 1 \leq d \leq n-1}\right)} d\right)$

$n = \prod_{\left(\substack{d|n \\ 1 \leq d \leq n-1}\right)} d$

$n = p^3$

$n = pq$

where p and q are prime numbers.

Problem 5 $f(n) = (-1)^n$

$(-1)^n = \sum_{\left(\substack{d|n \\ 1 \leq d \leq n-1}\right)} (-1)^d$

$1°\mathrm{n} = 2\mathrm{n} + 1;\ \mathrm{k} \in \mathbb{N}$

$-1 = \sum_{\left(\substack{d|n \\ 1 \leq d \leq n-1}\right)} (-1)^d$

We know that all the divisors of n are odd. So, they are all -1.

$-1 = (\tau(n) - 1).(-1) \Rightarrow \tau(n) = 2 \Rightarrow n = p,\ n \neq 2$

$2°\ n = 2^\alpha b;\ \alpha \in \mathbb{N},\ b = 2k + 1;\ k \in \mathbb{N}$

$1 = \alpha \cdot \tau(b).1 - 1 + \tau(b).(-1)$

$2 = (\alpha - 1).\tau(b)$

a)$\tau(b) = \alpha = 2$

$n = 4p$

b)$\alpha = 3;\ \tau(b) = 1;\ b = 1$

$n = 8$

Problem 6 $f(n) = \binom{2012}{n}$

We know that $\binom{a}{b} = \binom{a}{a-b}$. Therefore if n is a prime then 2012 must be $n + 1$ so $n = 2011$.

In general, $\binom{m}{n}$, if n is a prime then $m = n + 1$.

We have reasons to believe that this is the only solution.

Problem 7 $f(n) = F_n$

where F_n is n^{th} number of the Fibonacci sequence.

$\varphi = \frac{1+\sqrt{5}}{2},\ \psi = \frac{1-\sqrt{5}}{2}$.

$F_n = \frac{\varphi^n + \psi^n}{\sqrt{5}}$, and $F_n + F_{n+1} = F_{n+2}$

$F_{2n+1} \geq F_{2n} = \frac{\varphi^{2n}-\psi^{2n}}{\sqrt{5}} = \frac{\varphi^{n}-\psi^{n}}{\sqrt{5}}(\varphi^n + \psi^n)$
$\varphi^n + \psi^n \geq \frac{\varphi^{n}-\psi^{n}}{\sqrt{5}}$
$(\sqrt{5}-1)\cdot\varphi^n \geq (-\sqrt{5}-1)\cdot\psi^n \ /\cdot(-\frac{1}{2})$
$\varphi^{n-1} \geq \psi^{n-1} \ /\sqrt{5}$
$\frac{\varphi^{n-1}-\psi^{n-1}}{\sqrt{5}} \geq 0$
$F_{n-1} \geq 0$
$\Rightarrow F_{2n} \geq F_n^2$
$n = \left\lfloor \frac{a}{2} \right\rfloor$, $d|a$, $d < a \Rightarrow d \leq n$
$\sum_{d|a, 1\leq d\leq a-1} F_d \leq \sum_{i=1}^{n} F_i$
Proof that $\sum_{i=1}^{n} F_i < 2F_{n+1}$ follows:
Base: $n = 1$
$F_1 < 2F_2$
$1 < 2$
Assumption: For a natural number n $\sum_{i=1}^{n} F_i < 2F_{n+1}$
Step for $n+1$:
$\sum_{i=1}^{n+1} F_i < 2F_{n+2}$ $\Leftarrow$according to the assumption $\Leftarrow 3F_{n+1} \leq 2F_{n+2} \Leftrightarrow F_{n-1} \leq F_n$
Conclution: For every natural number n $\sum_{i=1}^{n} F_i < 2F_{n+1}$.
According to the proof above $\sum_{d|a,1\leq d\leq a-1} F_d < 3F_n$ which is less or equal to F_a
if $F_n \geq 3 \Leftrightarrow n \geq 4 \Rightarrow$If $a \geq 8$ a isn't $f-perfect$.
By manually checking remaining possibilities($a = 1, 2, ...7$) we came to a conclution that 2 is
the only f-perfect number.

Problem 8 A)$f(n) = \varphi(n)$
$\sum_{\left(\substack{d|n \\ 1\leq d\leq n-1}\right)} \varphi(d) = \varphi(m)$
$\sum_{\left(\substack{d|m \\ 1\leq d\leq m-1}\right)} \varphi(d) = \varphi(n)$
$\sum_{d|n} \varphi(d) = \varphi(m) + \varphi(n)$
$\sum_{d|m} \varphi(d) = \varphi(n) + \varphi(m)$
$m = n = 2.\varphi(m) = 2.\varphi(n)$
$n = 2.\varphi(n)$
According to the 2. task $n = 2^\alpha$
$(b)f(n) = \ln n$
$\sum_{\left(\substack{d|n \\ 1\leq d\leq n-1}\right)} \ln d = \ln m$
$\sum_{\left(\substack{d|m \\ 1\leq d\leq m-1}\right)} \ln(d) = \ln(n)$
$\sum_{d|n} \ln(d) = \ln(m) + \ln(n)$
$\sum_{d|m} \ln(d) = \ln(n) + \ln(m)$
$\sum_{d|m} \ln(d) = \sqrt{n^{\tau(n)}}$
$n^{\tau(\mathrm{n})} = m^2 n^2 = m^{\tau(\mathrm{m})}$
$m^2 = n^{\tau(\mathrm{n})-2}$
$n^2 = m^{\tau(\mathrm{m})-2}$

$1^{\circ}\tau(n)\geq 4$
$\tau(n)-2\geq 2$
$m^2\geq m^{\tau(\mathrm{m})-2}$
$\tau(m)\leq 4$
$2^{\circ}\tau(n)\leq 4$
Analogously $\tau(m)\geq 4$
$\tau(n)\leq 4, \tau(m)\geq 4$
$a)\tau(n)=1$
$n=1=m$,no solutions
$b)\tau(n)=2$
$m^2=n^{\tau(n)-2}$
$m^2=1=m=n$, no solutions
$c)\tau(n)=3$
$n=p^2$
$m^2=n\Rightarrow m=p\Rightarrow\tau(m)=2$, so no solutions
$d)\tau(n)=4$
$\mathrm{I}^{\circ}n=p^3$
$m^2=p^6\Rightarrow m=p^3$
So,the solutions are $m=n=p^3$.
Proof: $\ln 1+\ln p+\ln p^2=\ln p^3$
$1.p.p^2=p^3$
$\mathrm{II}^{\circ}n=pq$
$m^2=p^2q^2$
$m=pq$
So,the solutions are $m=n=pq$.
Proof:
$\ln 1+\ln p+\ln q=\ln pq$
$1.p.q=pq$