PROBLEM 8: DISCHARGED BATTERIES

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Abstract. In the following paper we investigate the minimum number of 2-tries needed to find two charged batteries among u uncharged and c charged. We also suggest a strategy for $u = 4$, $c = 4$ 3-try. We present a proof for the cases $u \geq c$. In the process of the research, we use purely combinatorial arguments and searches, obtained using computer calculations.

Definition 1. *Exhausting Strategies a way to prove the minimum of n-tries by exhausting all possible strategies in terms of how many times we have paired each battery. While using this method we will show counter-examples (a sequence of + and that fail the strategy).*

Definition 2. *Big Battery a battery we have paired more than once.*

Definition 3. *Big Pair a 2-pair between 2 big batteries.*

Definition 4. *Half-big battery - a battery paired with both big and not-big batteries.*

Lemma 1. *If the minimum of 2-tries needed to find a charged pair from U uncharged and C charged batteries is M, then the minimum of 2-tries needed to find a charged pair from U uncharged and $C - 1$ charged is not less than M and vice versa.*

Proof. Let the minimum of 2-tries for U and C be M and the minimum of 2-tries for U and $C - 1$ be less than M. We 'ban' a battery from the group of U and C and we are left with U and $C - 1$ (minimum less than M and M at the same time, contradiction) or $U - 1$ and C. In the second case we will consider a charged battery uncharged and we are in the first case. The same way, we can prove the vice versa. □

Problem 1. *A racer has an electric car that needs 2 batteries to run. Unfortunately, the only reserve at his disposal is a set of 10 batteries, and the only thing he knows is that 5 of them are charged and the other five are uncharged. Call a 2-try the act of putting two batteries in the car and trying to make it run. What strategy would you suggest to the racer to apply in order to minimize the number of 2-tries needed to find a charged pair?*

Solution: We will prove that the number of 2-tries needed to find a charged pair is 8. Strategy: $(b1 - b2); (b1 - b3); (b2 - b3); (b4 - b5); (b5 - b6); (b4 - b6); (b7 - b8); (b9 - b10)$. It is clear that using this strategy we can make the car run. Let the minimum of 2-tries be 7 (if it is less than 7 we will be able to make the car run with 7 tries). Let $(a1a10)$ be the number of times we have paired each battery $(b1b10)$. Let $1 \geq a1 \geq a2 \geq \cdots \geq a10$. If $a(k)$ equals zero, then according to our lemma the minimum of tries needed is at least 9 (If only $a1$ equals zero then we can assume we have $U = 5$, $C = 4$, where $M = 9$). During each 2-try we try 2 batteries so during 7 2-tries we have tried 14 batteries. Therefore $a1 + \cdots + a10 = 14$. If $a6 = 2$ then clearly an equality cannot hold. Therefore $a1 = \cdots = a6 = 1$ and $a7 + a8 + a9 + a10 = 8$. If $a7 = a8 = 1$, $a9 + a10 = 6$. There is a maximum of 1 big pair , therefore $a9$ and $a10$ are paired in a minimum of 5 pairs and therefore the maximum of not-big pairs is 2. A counter-example is $b9 = b10 = -$ and randomly placed minuses in the not-big pairs. If $a8 = 2$ (it clearly cannot be 3) , $a7 = 1$, $a9 + a10 = 5$, obviously $a9 = 2$, $a10 = 3$. The maximum of big pairs is 3. If the big pairs are 2 or less then the not-big pairs are 2 or less (7 pairs containing big batteries, 2 inner, therefore the 3 big batteries are paired in $7 - 2 = 5$ different pairs). A counter-example is the following: $b8 = b9 = b10 = -$ and random minuses in the 2 not-big pairs. Let the big pairs be 3. Obviously the only option is $(b10-b9), (b10-b8), (b9-b8), (b10-b7), (b6-b5), (b4-b3), (b2-b1)$ (In our solution if $a(i) = a(j)$ then $b(i)$ and $b(j)$ are indistinguishable). A counter-example is the following: $b10 = b9 = b6 = b4 = b2 = -$. In the last case, where $a7 = a8 = a9 = a10 = 2$ once again we exhaust all options $(1, 2, 3, 4$ big pairs). Therefore the minimum is at least 8 and an example-strategy is provided.

Problem 2. *Now, suppose that the car is bigger and it needs three charged batteries instead. Suppose also that the racer has only eight batteries, but still a symmetric distribution between charged and uncharged. Give a strategy to minimize the number of 3-tries.*

Solution: The minimum is 19. A strategy is 123, 124, 134, 234, 456, 457, 467, 567, 481, 482, 483, 485, 486, 125, 126, 135, 136, 235, 236.

Problem 3. *Investigate the case where there are $c > 2$ charged batteries and $u > 0$ uncharged ones. What is the order of growth of the number of 2-tries as a function of c and u? Is there anything special about the case $c = u$?*

Solution: Let $C = U > 6$ Strategy: $b_{2k+1} - b_{2k+2}$ for $k \in [0, U-1]$, $b_1 - b_3, b_2 - b_3, b_1 - b_4, b_2 - b_4$. Using this strategy we can get the car running in $U + 4$ 2-tries. Let the minimum be $U + 3$ then

$$\sum_{i=1}^{2U} b_i = 2U + 6$$

If b_{2U-6}=2, then $S > 2U+6 \Rightarrow b_1 = b_2 = \cdots = b_{2U-6} = 1$, $b_{2U-5} + b_{2U-4} + b_{2U-3} + b_{2U-2} + b_{2U-1} + b_{2U} = 12$ If $b_{2U-5} = b_{2U-4} = b_{2U-3} = 1$ our counter-example is the case where the last three batteries have a negative charge and the rest are alternatively different. If $b_{2U-5} = b_{2U-4} = 1$ If there are less than 4 big pairs the same counter-example still works. If there are 4 pairs we have 6 different cases for our strategy and in each one we will have a negative charge in each battery which is in a half-big pair and alternative charges in the rest. The last charge we will have in the big pair which consist of two big numbers. If there are 5 big pairs we have negative charges in the two batteries with 3 pairs and alternative placement for the rest. Analogically the last two cases are exhausted. We have proven that the minimum is $U - 4$. Let us study the case in which $U < C$. Let $C = U + 1$. Analogically the minimum is $U + 2$. The strategy is $b_{2k+1} - b_{2k+2}$ $k \in [0, U-1]$, $b_{2U+1} - b_1, b_{2U+1} - b_2$. Let $C = U + i$, $i \in [2, +\infty]$. The strategy is $b_{2k+1} - b_{2k+2}$ $k \in [0, U]$. We assume that we can start the car in U tries $\Rightarrow$ $b_1 + b_2 + \cdots + b_{2U+i} = 2U$. Obviously we have i batteries which we have not tried or $b_1 = b_2 = \cdots = b_i$=0 $\Rightarrow$ we reach the case in which we have $2U$ batteries in which the number of negative ones is at least U. We have already proven that in this one the number of tries needed is $U+4 \Rightarrow$ the minimum of tries is $U+1$. Let $U = C + i$ $i \in [0, 2/3U]$. If $2 \mid i$ $min = U + 4 + \frac{3}{2}i$, if $2 \nmid i$ $min = U + 5 + \frac{3}{2}(i-1)$.

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