PROBLEM 7: AN EXPERIMENT

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Abstract. In the following paper we consider the properties of n-tuples in $(\mathbb{F}_2)^n$. We approach different combinatorial and pure algebraic techniques to obtain answers to the given questions in the problem and to continue the research by investigating an amount of important questions in this topic. In addition we present our ideas of generalizing the subproblems that we have given partial answers to. In the paper one can find a partial exact answers (for $l = \{1, 2\}$) to **Question 7.1.a** and an formula, convenient for computer calculations for the case $l = 3$. Necessary and sufficient conditions to count a n-tuple for arbitrary l are given. We present a complete proof of **Question 7.1.b** and obtain a stronger statement. These results can be found in the first section of the paper. All of the propositions in **Question 7.1** are generalized as far as technically possible in all of the important directions. The authors believe that these new subproblems were unexplored before the execution of this research, due to the lack of information in internet, papers and mathematical literature. All of these new generalizations, results and comments are presented in the second section of the document. All of the conjectures and propositions are verified by computer programs.

1. Balanced n-tuples in $(\mathbb{F}_2)^n$. Question 7.1

1.1. **Maintaining balance after transformations. Question 7.1.a.** First we shall present the definitions and questions presented in the statement of the problem. We consider the set $\mathbb{F}_r$ as the set residues of modulo r with addition rule $x + y = (x + y) \mod r$, for $\{x, y\} \in \mathbb{F}_r$. We can easily see that this set with the addition rule is actually ring and later we can refer to its property.

Definition 1.1.1. *With α we denote the operation on the set of n-tuples, such that*

$$\alpha(x_0, x_1, \dots x_{n-1}) = (x_0 + x_1, x_1 + x_2, \dots, x_{n-1} + x_0)$$

for a given n-tuple $\in (\mathbb{F}_r)^n$.

Definition 1.1.2. *Let $\mu(x)$ denote the number of 0's in a given n-tuple $x = (x_0, x_1, \dots, x_{n-1})$, where $x \in (\mathbb{F}_2)^n$. We shall call x* balanced *if $\mu(x) = n/2$. We refer to a balanced x as a* super-balanced *n-tuple if every image $\alpha^i(x)$ is balanced for every $i \in \mathbb{N}$.*

Of course, from the definition follows that if a n-tuple is balanced, then n is even.

Definition 1.1.3. *Let $c_l(n)$ denote the number of balanced n-tuples $x \in (\mathbb{F}_2)^n$, where $\alpha^i(x)$ is balanced for all $1 \le i \le l$.*

Question 1.1.1. *What is the value of $c_l(n)$?*

We begin answering this question by setting some conventions and definitions before we present the propositions.

Definition 1.1.4. *We use the convention*

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1.5. *Let us associate to every n-tuple $\in (\mathbb{F}_2)^n$ the following form $\delta_0 = (a_0, b_0, \ldots, a_{k-1}, b_{k-1})$, where we have placed the n-tuple on a circle and a_i denotes the number of consecutive 1's in $(\mathbb{F}_2)^n$ in the i^{th} cluster and b_i denotes the number of consecutive 0's in the same ring in the next cluster. We shall say that a_i and b_i are the number of elements in* groups *of consecutive scalars.*

For example, let us consider a given sequence $(1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0)$. A δ_0 form of it is $(2, 1, 3, 1, 2, 1, 1, 1, 2, 3)$, where the set of the a's (starting from a_0 in increasing i order) is $\{2, 3, 2, 1, 2\}$. A δ_0 form of $(0, 0, 1, 1, 0, 1, 0)$ is $(2, 1, 1, 3)$.

We should note that in the case of the form δ_0, the beginning of the n-tuple is irrelevant due to symmetry. Furthermore, because of the fact that the n-tuple is placed on a circle, we have a fixed even number of groups. It is easy to see that if we have a balanced n-tuple then $\sum_{i=0}^{k-1} a_i = \sum_{i=0}^{k-1} b_i = n/2$.

Lemma 1.1.1. *Let us consider a given balanced n-tuple $x \in (\mathbb{F}_2)^n$. We have that $\alpha(x)$ is balanced, if and only if x is of the form $\delta_0 = \big(a_0, b_0, \ldots, a_{n/4-1}, b_{n/4-1}\big)$.*

Proof. Suppose that x is of the form $\delta_0 = (a_0, b_0, \ldots, a_{k-1}, b_{k-1})$. Let us consider a maximal group of r consecutive equal elements. We have that after the transformation it maps into a group of $r-1$ 0's and a group of a single 1 (Note that r can be equal to 1). Hence,

$$\mu(\alpha(x)) = \sum_{i=0}^{k-1} (a_i + b_i) - 2k = n - 2k. \tag{1.1}$$

We have that $\alpha(x)$ is balanced which is equivalent to $\mu(\alpha(x)) = n/2$. The positioning of the 1's is irrelevant. Now, using (1.1) we have that $k = n/4$, which completes the proof. ∎

For example, in $(0, 0, 1, 1, 1, 0)$ a maximal group of consecutive elements is the subset $(1, 1, 1)$.

Furthermore, because k is a positive integer we have that $c_l(n) = 0$ if n is nondivisible by 4 and from now on we shall consider only n, multiples of 4. Let us consider a n-tuple x of the form δ_0. We can either set a beginning of x at the space between two adjacent groups of different elements or between the elements of a single group. Hence, if we consider linear n-tuples (the ones we are interested in the problem), we have two cases for the δ-forms.

Definition 1.1.6. *We can associate to every n-tuple $\in (\mathbb{F}_2)^n$ one of the following forms: $\delta_1 = (a_0, b_0, \ldots, a_{k-1}, b_{k-1})$ or $\delta_2 = (a_0, b_0, \ldots, a_{k-1}, b_{k-1}, a_k)$, where a_i denotes the number of consecutive 1's in $(\mathbb{F}_2)^n$ in the i^{th} cluster and b_i denotes the number of consecutive 0's in the same ring in the next cluster.*

One can easily see that the beginning of a n-tuple can be zero. Due to symmetry in most of the cases we need to consider only the given forms. In case we need to consider a n-tuple that starts with 1 as a separate case, we shall only change the places of the a's and the b's.

For example, $(1,0,0,1,1,0,1,0,0,0,1,1)$ has δ_2 form $(1,2,2,1,1,3,2)$. The sequence $(0,0,1,0)$ has a δ_2 form of type (b_0,a_0,b_1), where $b_0=2$, $a_0=1$ and $b_1=1$.

Once again, if we have a balanced n-tuple, from Lemma (1.1.1) $k=n/4$. Before we consider $c_l(n)$ we are going to present the following lemma.

Lemma 1.1.2. *The number of solutions of*

$$\sum_{i=0}^{m-1} x_i = S,$$

where S is a fixed integer constant, x_i is a nonnegative integer for $0\le i\le m-1$ and $x_i\ge s_i$, where s_i is a fixed nonnegative integer constant for $0\le i\le m-1$ is

$$\binom{S-\sum_{i=0}^{m-1} s_i+m-1}{m-1}.$$

Proof. Let us consider the m-tuples $X=(x_1,x_2\dots,x_n)$ and $S=(s_1,s_2\dots,s_n)$. Let $Y=(y_1,y_2,\dots,y_n)=X-S$. Then, its sufficient to calculate the number of the solutions of

$$\sum_{i=0}^{m-1} y_i = S-\sum_{i=0}^{m-1} s_i \tag{1.2}$$

We should note that $y_i\ge 0$.

Now lets us consider the set T of $\left(S-\sum_{i=0}^{m-1} s_i+m-1\right)$-tuples with elements in $\mathbb{F}_2$ with exactly $m-1$ ones and the set $S^{'}$ of m-tuples of solutions of (1.2). Let us take $t\in T$, such that the indexes of the positions of the ones are respectively $0\le t_0<t_1<\dots<t_{m-2}$ and consider the function $f:T\to S^{'}$ that maps t on $(t_0,t_1-t_0-1,\dots,t_{m-2}-t_{m-3}-1)$, then we have that f is a bijection. Now

$$|T|=\binom{S-\sum_{i=0}^{m-1} s_i+m-1}{m-1}$$

and the proof is completed. □

Note that the m-tuples in Y are called disordered nonnegative m-partitions of $S-\sum_{i=0}^{m-1} s_i$. See [1]. We should also state that we would know some of the elements x_i. In that case we just remove them from the total sum and lower the number of elements in the equation.

Proposition 1.1.1. *The following is true:*

$$c_1(n)=\binom{n/2}{n/4}^2$$

Proof. Let us consider the number of n-tuples $c_{1,1}(n)$ of the form
$\delta_1=(a_0,b_0,\dots,a_{k-1},b_{k-1})$ and $c_{1,2}(n)$ of the form $\delta_2=(a_0,b_0,\dots,a_{k-1},b_{k-1},a_k)$ with the required properties. We have that the first element in these tuples is 1. Hance, due to symmetry we have that

$$c_1(n)=2\left(c_{1,1}(n)+c_{1,2}(n)\right).$$

First, we shall derive a formula for $c_{1,1}(n)$. From Lemma (1.1.1) we have that $k=n/4$. Hence, the necessary and sufficient conditions are

$$\begin{cases}\sum_{i=0}^{n/4-1} a_i=n/2,\\ \sum_{i=0}^{n/4-1} b_i=n/2.\end{cases}.$$

We should note that $a_i \geq 1$ and $b_i \geq 1$. Now, using Lemma (1.1.2), we have that the number of solutions of $\sum_{i=0}^{n/4-1} a_i = n/2$ which is the same as the one of $\sum_{i=0}^{n/4-1} b_i = n/2$ is $\binom{n/2-1}{n/4-1}$. Thus,

$$c_{1,1}(n) = \binom{n/2-1}{n/4-1}^2. \tag{1.3}$$

Now, we shall consider $c_{1,2}(n)$. We have that the number of the a-elements is $n/4+1$ and the number of the b-elements is $n/4$. Hence, we face the following conditions

$$\begin{cases} \sum_{i=0}^{n/4} a_i = n/2, \\ \sum_{i=0}^{n/4-1} b_i = n/2. \end{cases}.$$

Thus, from Lemma (1.1.2) we have that

$$c_{1,2}(n) = \binom{n/2-1}{n/4}\binom{n/2-1}{n/4-1} = \binom{n/2-1}{n/4-1}^2. \tag{1.4}$$

Now, from 1.3 and 1.4 we have that

$$c_1(n) = 4\binom{n/2-1}{n/4-1}^2 = \left(2\binom{n/2-1}{n/4}\right)^2.$$

We state that

$$2\binom{n/2-1}{n/4} = 2\frac{(n/2-1)!(n/4)}{(n/4-1)!(n/4)!(n/4)} = \frac{(n/2-1)!(n/2)}{((n/4)!)^2} = \binom{n/2}{n/4}$$

to complete the proof. □

Proposition 1.1.2. *The following is true:*

$$c_2(n) = 4\sum_{i=1}^{n/4-1} \binom{n/4}{i}^2 \binom{n/4-1}{i-1}\binom{n/4-1}{i}$$

Proof. We have that x, $\alpha(x)$ and $\alpha^2(x)$ are balanced. From Lemma (1.1.1) we have that the necessary and sufficient condition for $\alpha^2(x)$ to be balanced is $k = n/4$ if $\alpha(x)$ is of the form δ_i, for $i = 1, 2$. Hence, after the transformation of x, we should obtain a n-tuple with $k = n/4$. Lemma (1.1.1) states that if x is of the form δ_i, $i = 1$, 2, then $k = n/4$. Now, let us consider x as the form $\delta_0 = (a_0, b_0, \dots, a_{k-1}, b_{k-1})$. Every group with more than one elements, after the transformation, is shorter, but remains. If we consider a group of length one, then after the transformation it is deleted. We should note that between two groups of length greater than one we obtain a new group. Now, let us have exactly r groups of length one in $x = \delta_0$. From the above considerations, we have the following equation:

$$2\frac{n}{2} - 2r = \frac{n}{2}$$

Thus, $r = n/4$ and it also is a sufficient condition. Let us denote with $c_{2,1}(n)$ the number of n-tuples with the properties, we are looking for, of the form δ_1 and respectively with $c_{2,2}(n)$ - the number of these of the form δ_2. First we shall calculate $c_{2,1}(n)$. We have the following conditions:

$$\begin{cases} \sum_{i=0}^{n/4-1} a_i = n/2, \\ \sum_{i=0}^{n/4-1} b_i = n/2, \\ n/4 \text{ distributed elements of length one.} \end{cases}$$

Let us have t elements of length one among all a_i, $\forall\ i$. Then, we have $n/4-t$ elements, distributed among the b-elements. There are $\binom{n/4}{t}\binom{n/4}{n/4-t} = \binom{n/4}{t}^2$ ways of placing the 'ones'. Now, using Lemma (1.1.2), we have that

$$c_{2,1}(n) = \sum_{i=1}^{n/4-1} \binom{n/4}{i}^2 \binom{n/4-1}{i-1}\binom{n/4-1}{i}.$$

We continue with deriving the formula for $c_{2,2}(n)$. We have the following conditions:

$$\begin{cases} \sum_{i=0}^{n/4} a_i = n/2, \\ \sum_{i=0}^{n/4-1} b_i = n/2, \\ n/4 \text{ distributed elements of length one, except } a_0 \text{ and } a_{n/4}. \end{cases}$$

Once again let us have t elements with length one among the a-elements. Hence, there are $\binom{n/4-1}{t}\binom{n/4}{n/4-t} = \binom{n/4-1}{t}\binom{n/4}{t}$ ways of placing the 'ones'. Now, using Lemma (1.1.2) we have that

$$c_{2,2}(n) = \sum_{i=1}^{n/4-1} X_i Y_i = c_{1,2}(n),$$

where

$$X_i = \binom{n/4-1}{i}\binom{n/2-i-(n/4+1-i-2)-1}{n/4+1-i-1}$$

and

$$Y_i = \binom{n/4}{n/4-i}\binom{n/2-(n/4-i)-i-1}{i-1}.$$

Now, due to symmetry $c_2(n) = 2\left(c_{2,1}(n)+c_{2,2}(n)\right) = 4c_{2,1}(n)$ and the proof is completed. □

Now, we shall concentrate on finding a formula for the case of $l=3$.

Lemma 1.1.3. *The necessary and sufficient conditions for a cyclic balanced n-tuple (considered in δ_0) to have balanced images until $l=3$ is*

$$\begin{cases} \sum_{i=0}^{n/4-1} a_i = n/2, \\ \sum_{i=0}^{n/4-1} b_i = n/2, \\ n/4 \textit{ distributed elements of length 1.} \\ n/4 \textit{ number of elements} = 2 + \textit{number of pairs of adjacent element, both of size 1} \end{cases}$$

Proof. We have that x, $\alpha(x)$, $\alpha^2(x)$ and $\alpha^3(x)$ are balanced. $\alpha^3(x)$ is balanced, if and only if $\alpha(x)$ requires the conditions for $l=2$. We have that x should require the conditions for $l=2$, because $\alpha^2(x)$ is balanced. Now, we want to obtain exactly $n/4$ elements of size one after the transformation of x. We can obtain an element of size one only from one of size two or of two adjacent elements, greater than one. Let us have k element of size two. Hence, we have $n/4-k$ number of pairs of adjacent element, greater than one. Because we are working on a circle, we have a total number of $n/2$ adjacent elements. Thus, the adjacencies where we have an element of size one are exactly $n/2 - n/4 - k = n/4 + k$. For a moment, let us suppose that every element of size one generates exactly two adjacencies with another two elements of size one. So, we have a total of $n/4 \times 2 = n/2$ imagined adjacencies. But two adjacent ones will actually generate with one less adjacency. Hence, the number of adjacent ones is $n/2-(n/4+k) = n/4-k$ and this completes the proof. □

Let us consider an n-tuple x and its representation δ_1. Suppose that we have exactly k_1 pairs of adjacent elements of size one of the form (a_ib_i), meaning that $a_i = b_i = 1$, k_2 adjacent pairs of elements of the form (b_ia_{i+1}), where $a_i = b_i = a_{i+1} = b_{i+1} = 1$, meaning that we have k_2 adjacent pairs of adjacent elements of size one of the form (a_ib_i). Also, let us have k_{3b} adjacent pairs of elements of the form (b_ia_{i+1}), where $b_i = a_{i+1} = 1$, $a_i \neq 1$ and $b_{i+1} = 1$, meaning that we have an adjacent element of size one to the right of a pair of adjacent 'ones' of the first type and analogously we consider k_{3a} adjacent pairs of elements of the form (b_ia_{i+1}), where $b_i = a_{i+1} = 1$, $a_i = 1$ and $b_{i+1} \neq 1$. Let there be k_4 adjacent pairs of elements of the form (b_ia_{i+1}), where $b_i = a_{i+1}$, $a_i \neq 1$ and $b_{i+1} \neq 1$. We shall have k_{5a} elements in the a's of size 1, non-adjacent with another element of size one. Analogously we consider k_{5b} elements in the b's of size 1, non-adjacent with another element of size one.

For example, if $\delta_1 = (2, 1, 1, 1, 5, 1, 6, 7, 2, 3, 1, 1, 1, 1)$, then $k_1 = 3$, $k_2 = 1$, $k_{3a} = 0$, $k_{3b} = 1$, $k_4 = 0$, $k_{5b} = 1$ and $k_{5a} = 0$.

Proposition 1.1.3. *The number of cyclic sequences (of the form δ_1) for $l = 3$ is exactly*

$$2\sum_{x_i} V_{k_{1i},k_{2i},k_{3ai},k_{3bi},k_{4i},k_{5ai},k_{5bi}}(n/4),$$

where x_i is over all of the sequences with the upper properties for fixed parameters k_{1i}, k_{2i}, k_{3ai}, k_{3bi}, k_{4i}, k_{5ai}, k_{5bi},

$$k_{3ai} = n/4 - 2k_{1i} - k_{3bi} - k_{5ai} - k_{5bi} - 2k_4i$$

and

$$V_{k_{1i},k_{2i},k_{3ai},k_{3bi},k_{4i},k_{5ai},k_{5bi}}(n/4) = \sum_{j=0}^{n/4-k_{1i}-k_{2i}-k_{3ai}-k_{3bi}-k_{4i}} X_{1j}X_{2j}X_{3j}X_{4j},$$

where $X_{1j} = \binom{n/4-k_{1i}-k_{3ai}-k_{5ai}-k_{4i}}{j}$, $X_{2j} = \binom{k_{1i}+k_{3ai}+k_{5ai}+k_{4i}-1}{n/4-k_{1i}-k_{3ai}-k_{5ai}-k_{4i}-j-1}$, $X_{3j} = \binom{n/4-k_{1i}-k_{3bi}-k_{5bi}-k_{4i}}{k_{2i}+k_{3ai}+j-k_{5bi}}$ *and* $X_{4j} = \binom{k_{1i}+k_{3bi}+k_{4i}+k_{5bi}-1}{k_{2i}+k_{3ai}-k_{5bi}+j-1}$.

Proof. We shall count the number of cyclic sequences $c_3^{''}(n)$ of the form δ_1. Due to symmetry we have that $c_3^{'}(n) = 2c_3^{''}(n)$. Now, let us consider the form

$$(a_0, b_0, \dots, a_{k-1}, b_{k-1}).$$

For the sake of apprehension, we shall repeat the characterization of x. Suppose that we have exactly k_1 pairs of adjacent elements of size one of the form (a_ib_i), meaning that $a_i = b_i = 1$, k_2 adjacent pairs of elements of the form (b_ia_{i+1}), where $a_i = b_i = a_{i+1} = b_{i+1} = 1$, meaning that we have k_2 adjacent pairs of adjacent elements of size one of the form (a_ib_i). Also, let us have k_{3b} adjacent pairs of elements of the form (b_ia_{i+1}), where $b_i = a_{i+1} = 1$, $a_i \neq 1$ and $b_{i+1} = 1$, meaning that we have an adjacent element of size one to the right of a pair of adjacent 'ones' of the first type and analogously we consider k_{3a} adjacent pairs of elements of the form $(b_i a_{i+1})$, where $b_i = a_{i+1} = 1$, $a_i = 1$ and $b_{i+1} \neq 1$. Let there be k_4 adjacent pairs of elements of the form $(b_i a_{i+1})$, where $b_i = a_{i+1}$, $a_i \neq 1$ and $b_{i+1} \neq 1$. We shall have k_{5a} elements in the a's of size 1, non-adjacent with another element of size one. Analogously we consider k_{5b} elements in the b's of size 1, non-adjacent with another element of size one.

By definitions and the requirement for the number of 'ones' we have that the number of the elements of size one is

$$2k_1 + k_{3a} + k_{3b} + k_{5a} + k_{5b} + 2k_4 = n/4. \tag{1.5}$$

Again by definitions we have exactly

$$k_1 + k_2 + k_{3a} + k_{3b} + k_4 \leq n/4 \tag{1.6}$$

number of pairs of adjacent elements of size one. Considering again the definitions we note that we have

$$k_1 + k_{3a} + k_{5a} + k_4 \tag{1.7}$$

elements of size one, among the a-elements and

$$k_1 + k_{3b} + k_{5b} + k_4 \tag{1.8}$$

elements of size one, among the b-elements. From the requirement of the number of elements of size two (Lemma (1.1.3)) and the definitions we have that it is exactly

$$n/4 - k_1 - k_2 - k_{3a} - k_{3b} - k_4.$$

Now, suppose that we have i elements of size two among the a-elements. There are exactly

$$\binom{n/4 - k_1 - k_{3a} - k_{5a} - k_4}{i} \tag{1.9}$$

ways of positioning them. Now, considering only the a's , (1.7) and their sum, we face with $n/4 - k_1 - k_{3a} - k_{5a} - k_4 - i$ elements, greater than 2, with sum $n/2 - k_1 - k_{3a} - k_{5a} - k_4 - 2i$. Using Lemma (1.1.2) we have that the number of solutions is

$$\binom{k_1 + k_{3a} + k_{5a} + k_4 - 1}{n/4 - k_1 - k_{3a} - k_{5a} - k_4 - i - 1}. \tag{1.10}$$

There are $n/4 - k_1 - k_2 - k_{3a} - k_{3b} - k_4 - i$ elements of size two among the b-elements. There are exactly

$$\binom{n/4 - k_1 - k_{3b} - k_{5b} - k_4}{k_2 + k_{3a} + i - k_{5b}}. \tag{1.11}$$

ways of placing them. Now, from (1.8) and the last argument we have $k_2 + k_{3a} + i - k_{5b}$ elements, greater than 2, with sum $k_1 + 2k_2 + 2k_{3a} + k_{3b} + k_4 - k_{5b} + 2i$. Thus, the number of solutions is

$$\binom{k_1 + k_{3b} + k_4 + k_{5b} - 1}{k_2 + k_{3a} - k_{5b} + i - 1}. \tag{1.12}$$

From (1.5) and (1.6) we find the ranges for the parameters in the sums in the formula. Combining (1.9), (1.10), (1.11), (1.12) we complete the proof. □

1.2. **Existence of super-balanced n-tuples. Question 7.1.b.** The answer to the question of existence of super-balanced n-tuples is negative. We are going to present a recursive represented necessary and sufficient conditions for a n-tuple to be counted in $c_l(n)$. We shall note that it is sufficient to prove this statement only for cyclic n-tuples, considered only in the form δ_1. Indeed, let us suppose that we have a super-balanced n-tuple x, represented in the form $\delta_2 = (a_0, b_0, \dots, a_{k-1}, b_{k-1}, a_k)$. But, the n-tuple, obtained from x, but considered with a_k as a beginning has the form $\delta_1 = (a_0 + a_k, b_0 \dots a_{k-1} b_{k-1})$ and maintains the required property. Finally, it is easy to see that if we have a super-balanced n-tuple, starting with 0, then we have a super-balanced one, starting with 1 (swapping the places of 1 and 0). Hence, to the end of the section we shall consider only n-tuples of the form δ_1.

Definition 1.2.1. *Let us consider the sequence*

$$\sigma = ((*_r k_r), (*_{r-1} k_{r-1}), \dots, (*_1 k_1), (*_0 k_0)),$$

where $*_i := \{=, \geq\}$. *We shall consider* σ *as a sub-consequence in* δ_0, *where* $*_{i-1} k_{i-1}$ *represents the size of the* i^{th} *consecutive element (from left to right). We shall use these sequences as* conditions, *meaning that the* n*-tuple, we are considering should possess an amount of these sub-consequences in its representation as* δ_1. *Let* $D_x(\sigma)$ *denotes the number of the conditions* σ *in the representation of* x *as* δ_1.

For example, let us consider a δ_1 form $(0, 2, 2, 3, 0, 5, 2)$. The number of conditions $((\geq 2), (= 2))$ are 2. Indeed the subsets $(2, 2)$ and $(5, 2)$ are of the form $((\geq 2), (= 2))$. An example for a condition of the form $((\geq 2), (= 2), (\geq 2))$ is the subsequence $(2, 2, 3)$.

Definition 1.2.2. *Let us denote with* $C_l(n)$ *all the requirements that a* n*-tuple of the form* δ_1 *should hold to be counted in* $c_l(n)$. *We shall consider with* $C_l^{'}(n)$ *the new requirements that we face in* $C_l(n)$ *(the ones different from* C_{l-1}*), where* $l \geq 2$.

For example, we have already considered $C_1(n)$, $C_2(n)$ and $C_3(n)$ in obtaining formulas for the corresponding numbers. Indeed $C_2^{'}(n) = n/4$ distributed elements of size one, $C_3^{'}(n) = n/4$ number of elements $= 2+$ number of pairs of adjacent element, both of size 1. It turns out, that we can easy represent these requirements with the upper-defined conditions.

Proposition 1.2.1. *The following is true:*

$$C_l(n) := \begin{cases} C_{l-1}(n), \\ C_l^{'}(n) := \textit{After the transformation to obtain } C_{l-1}^{'}(n), \end{cases}$$

for $l \geq 2$.

Proof. We have that $x, \alpha(x), \alpha^2(x), \dots, \alpha^l(x)$ should be balanced. Suppose that we have obtained $C_{l-1}(n)$ and respectively $C_l^{'}(n)$. From the definitions we have that the necessary and sufficient condition for $\alpha^l(x)$ to be balanced is $\alpha(x)$ to require $C_{l-1}(n)$. Hence, after the transformation of x we should obtain $C_l^{'}(n)$. On the other hand x requires $C_{l-1}(n)$ also, because $x, \alpha(x), \alpha^2(x), \dots \alpha^{l-1}(x)$ are balanced and we complete the proof. □

We have that $C_l^{'}(n)$ is a set of conditions. We would like to know an algorithm that gives us the conditions, that after a transformation lead to a given condition. From the definition follows that we can always end with a given condition after transformation. Also, it's obvious that a given condition should come from a connected one. In that way we would know how to generate $C_l^{'}(n)$ from a fixed one $C_{l-1}^{'}(n)$. Before we proceed with it, we should note some properties of the n-tuples in $(\mathbb{F}_2)^n$:

Claim 1.2.1. *Let us consider the form δ_1 and the condition $(=t)$ in it. After a transformation we can obtain $(=t)$ only from $(=(t+1))$ (we shall say* WAY1*) or $((\geq 2),(=1),\dots,(=1),(\geq 2))$ (*WAY2*), where we have $t-1$ $(=1)$'s. Analogously we have that we can obtain (≥ 1) only from $(\geq (t+1))$ or $((\geq 2),(=1),\dots,(=1),(\geq 2))$, where we have* **more** *than $t-2$ $(=1)$'s. In a conclusion,* ***we can obtain a given group only in 2 ways****.*

Claim 1.2.2. *Let us consider the form δ_1 and the condition $\sigma = ((*_1k_1),(*_2k_2))$. Let σ_1 be a* parent *of σ, i.e. after a transformation it leads to σ. Suppose that both $(*_1k_1)$ and $(*_2k_2)$ are obtained in $WAY1$, i.e., they come from $((*_1(k_1+1)),(*_2(k_2+1))$. But we have two adjacent elements, bigger than 2. So, we have a new group between them after the transformation, which is a contradiction with the number of the groups. Hence, if $(*_2k_2)$ comes from WAY1, then $(*_1k_1)$ comes from WAY2. Suppose that both $(*_1k_1)$ and $(*_2k_2)$ are obtained in $WAY2$. But $WAY2$ for $(*_2k_2)$ requires an element (≥ 2) adjacent to the left, which is a contradiction with $WAY2$ for $(*_1k_1)$. Hence, if $(*_2k_2)$ comes from WAY2, then $(*_1k_1)$ comes from WAY1. In conclusion,* ***a parent is strictly determined from the way of obtaining the last group in the condition****.*

Algorithm 1 Finds the parents p_1 and p_2 of $\sigma = ((*_rk_r),\dots,(*_0k_0))$

```
p_1 ← (*_0(k_0 + 1))                                  ▷ Initialize the first parent
for i = 1 → r do
    if i is odd then                                  ▷ The determined ways
        p_1 := (= 1) *_i(k_i−1)... (= 1) + p_1        ▷ *_i(k_i − 1) (= 1)'s added
    else
        p_1 := (*_i(k_i + 1)) + p_1                   ▷ Even i
    end if
end for
if r is odd then
    p_1 := (≥ 2) + p_1                                ▷ Necessary condition in WAY2
end if
p_2 ← ((= 1), *_0(k_0−1)..., (= 1), (≥ 2))            ▷ Initialize p_2. *_0(k_0 − 1) (= 1)'s added
for i = 1 → r do
    if i is even then                                 ▷ The determined ways. Swapped parity.
        p_1 := (= 1) *_i(k_i−1)... (= 1) + p_1        ▷ *_i(k_i − 1) (= 1)'s added
    else
        p_1 := (*_i(k_i + 1)) + p_1                   ▷ Even i
    end if
end for
if r is even then
    p_1 := (≥ 2) + p_1                                ▷ Necessary condition in WAY2
end if
    return p_1 and p_2                                ▷ The constructed parents
```

For example, the parents of $(= 3)$, obtained using the algorithm, are $(= 4)$ and $((\geq 2), (= 1), (= 1), (\geq 2))$.

Corollary 1.2.1. *The number of new conditions for the case of l transformations, where $l >= 2$ is exactly 2^{l-2}. In other words, $|C_l^{'}(n)| = 2^{l-2}$.*

Proof. We proceed by induction. For $l = 2$ we have $1 = 2^{2-2}$ new condition. From the above algorithm, we have that we double the number of conditions with every new transformation. Let us assume, that for $l = k$ we have 2^{k-2} conditions. Then, for $k+1$ we obtain exactly $2 \times 2^{k-2}$ new conditions. The last observation, completes the induction and respectively the proof. □

The upper results give a fully classification of the necessary and sufficient requirements a n-tuple to be counted in $c_l(n)$. Now, from Dirichlet's principle we can see that if n is finite, then $\exists L_x$, such that $\alpha^{L_x} x = x$, for a given balanced x. The requirements $C_l(n)$ are valid only if $l \geq L_x$. We shall give a bound on L_x, which we shall later use in the proof of nonexistence.

Lemma 1.2.1. *The following is true:*

$$L_x \geq n,$$

for $x \in (\mathbb{F}_2)^n$.

Proof. Let us consider a given n-tuple $(\mathbb{F}_2)^n \supseteq x = (x_{n-1}, x_{n-2}, \dots, x_0)$. Let us associate to x a polynomial $P_x(t)$ defined in the following way:

$$P_x(t) = \sum_{i=0}^{n-1} x_i t^i$$

We should note that if we consider a given polynomial $Q(t)$ with $degQ = m > n$ and coefficients in $\mathbb{F}_2$, we will set $x^i = x^{i \mod n}$, where $i = n+1, \dots, m$. Given the following settings we have that if $\alpha(x) = x^{'}$, then $P_{x'}(t) = P_x(t)(1+t)$. By definition we have that $x = \alpha^{L_x}(x)$, i.e. $P_x(t) = P_{\alpha^{L_x}(x)}(t)$. But $P_{\alpha^{L_x}(x)}(t) = P_x(t)(1+t)^{L_x}$. Hence, we face the following equation

$$P_x(t)(1+t)^{L_x} = P_x(t) \Leftrightarrow P_x(t)((1+t)^{L_x} + 1) = 0$$

We should note that $(1+t)^{L_x} + 1 = (1+t)^{L_x} - 1$ in the ring $\mathbb{F}_2$. Solving it for $P_x(t)$ leads us to $P_x(t) = 0$ or $(1+t)^{L_x} = 1$. In the first case we do not face a balanced n-tuple. The second case does not have a solution for $L_x < n$ (simple observation after expansion). The last argument completes the proof. □

Using the algorithm we can obtain necessary and sufficient conditions for an arbitrary l, including its smaller values.

Corollary 1.2.2. *The following is true:*

$$C_1(n) := \begin{cases} \sum_{i=0}^{n/4-1} a_i = n/2, \\ \sum_{i=0}^{n/4-1} b_i = n/2. \end{cases}$$

Corollary 1.2.3. *The following is true:*

$$C_2(n) := \begin{cases} C_1(n), \\ n/4 = D_x((= 1)). \end{cases}$$

Corollary 1.2.4. *The following is true:*

$$C_3(n) := \begin{cases} C_2(n), \\ n/4 = D_x((=2)) + D_x((\geq 2), (\geq 2)). \end{cases}$$

Corollary 1.2.5. *The following is true:*

$$C_4(n) := \begin{cases} C_3(n), \\ n/4 = sum\ of \begin{cases} D_x((=3)), D_x((\geq 2), (=1), (\geq 2)), \\ D_x((\geq 2), (=1), \overset{\geq 1}{\ldots}, (=1), (\geq 3)), \\ D_x((\geq 3), (=1), \overset{\geq 1}{\ldots}, (=1), (\geq 2)). \end{cases} \end{cases}$$

Now we are ready to prove the main result.

Proposition 1.2.2. *Super-balanced n-tuples do not exist.*

Proof. Let us consider an arbitrary chosen condition $\sigma = ((*_r k_r), \ldots, (*_0 k_0))$ and let us denote with p_1 and p_2 its parents (as noted in the algorithm). Let us denote with $h(x)$ the sum of the k's in the groups of a given condition x.

First we shall prove that $h(p_i) > h(\sigma)$. Because of the determination of the 0^{th} group, we have that we use WAY1 in even indexes. Now let us consider a sub-condition of the following type - $((*_{2t+1} k_{2t+1}), (*_{2t} k_{2t}))$. According to the algorithm, we replace this sub-condition with sub-condition θ, such that $h(\theta) = k_{2t+1} - 1 + k_{2t} + 1 = k_{2t+1} + k_{2t}$. If r is odd, then we will finish with WAY2 and it will require (≥ 2). Hence, $h(p_1) = 2 + h(\sigma)$. If r is even, then we will finish with WAY1 and $h(p_1) = 1 + h(\sigma)$. In both cases $h(p_1) > h(\sigma)$.

For the case of p_2 we have that we use WAY1 in the odd indexes. We have that the 0^{th} group transforms into a one with $h(((=1), \overset{*_0(k_0-1)}{\ldots}, (=1), (\geq 2))) = 1 + h((*_0 k_0))$. Now let us consider a sub-condition of the following type - $((*_{2t} k_{2t}), (*_{2t-1} k_{2t-1}))$. According to the algorithm, we replace this sub-condition with sub-condition θ, such that $h(\theta) = k_{2t} - 1 + k_{2t-1} + 1 = k_{2t} + k_{2t-1}$ and the sum of this sub-condition does not change. Now, if r is odd, then we will finish with WAY1 and that will result $h(p_2) = 2 + h(\sigma)$. In r is even we will finish with WAY2 and we would have $h(p_2) = 3 + h(\sigma)$. In both cases $h(p_2) > h(\sigma)$. Hence,

$$h(p_i) > h(\sigma),$$

for an arbitrary chosen σ.

Now, let us consider $C_4^{'}(n)$.

The condition $(=3)$ is generated from $(=4)$ and $((\geq 2), (=1), (=1), (\geq 2))$, with sum 6.

The condition $((\geq 2), (=1), (\geq 2))$ is resulted from $((\geq 2), (=1), \overset{\geq 1}{\ldots}, (=1), (=2), (=1), \overset{\geq 1}{\ldots}, (=1), (\geq 2))$, with sum of the elements ≥ 8 or from $((\geq 3), (\geq 3))$. $((\geq 3), (\geq 3))$ on the other hand has parents $((\geq 2), (=1), (=1), (\geq 4))$ and $((\geq 4), (=1), (=1), (\geq 2))$, both having sum 8.

The conditions $((\geq 2), (=1), \overset{\geq 1}{\ldots}, (=1), (\geq 3))$ and $((\geq 3), (=1), \overset{\geq 1}{\ldots}, (=1), (\geq 2))$ both have sum ≥ 6.

The last observations imply that for $l = 6$ we have that $h(\sigma) \geq 8$ for every $\sigma \in C_6^{'}(n)$, such that $\sigma \neq (=5)$. Now, using simple induction on this basis we prove that $h(\sigma) \geq l + 2$ for every $\sigma \in C_l^{'}(n)$, such that $\sigma \neq (= l - 1)$, for $l \geq 6$.

From Lemma (1.2.1) is it sufficient to prove that $c_{n-1}(n) = 0$. We shall consider $C_{n-1}(n)$ and more specifically $C_{n-1}^{'}(n)$. We can see that $(= (n-2)) \in C_{n-1}^{'}(n)$.

But that means, we should have element, equal to $n-2$ among elements with sum $n/2$ ($n \geq 8$). So, $(=(n-2))$ does not exist. Now we take an arbitrary $\sigma \neq (=(n-2)) \in C'_{n-1}(n)$. If the number of the groups in σ is bigger than $n/2$, then it is impossible to have σ in a balanced n-tuple (See Lemma (1.1.1)). Suppose we have number of groups $\leq n/2$. We have that $h(\sigma) \geq n+1$, meaning that we have elements with sum $\geq (n+1)/2$ among elements with sum $n/2$. Contradiction with existence of such conditions. Hence, we cannot obtain $C'_{n-2}(n)$, which leads us to $c_{n-1}(n) = 0$, for $n \geq 8$.

We should consider only the case $n = 4$ now. We take a balanced 4-tuple. Let us consider the representation in the form δ_1. From Lemma (1.1.1) we have that it is (b_0, a_0), where $b_0 = 2$ and $a_0 = 2$. Hence,

$$D_x((=2)) + D_x(((\geq 2), (\geq 2))) = 3,$$

which is a contradiction with $C_3(n)$. Hence, $c_3(n) = 0$ for $n = 4$ to complete the proof.

□

The authors believe that this technique may be used in proving the following

Conjecture 1.2.1. *We have that $c_k(4k) > 0$ and $c_{k+1}(4k) = 0$.*

Summary 1.2.1. *In the section, we presented exact formulas for $l = 1$ and $l = 2$. For $l = 3$ we present a formula, convenient for computer calculations. We presented necessary and sufficient conditions, recurrent defined, for every l. Judging of the number of new conditions for an arbitrary l, the authors believe that a closed formula for $l \geq 4$ does not exist. Based on the fully classification of the conditions, we answered completely the question of existence of super-balanced n-tuples and presented a conjecture, which may lead to a powerful result.*

2. Generalizations, new questions and comments. Question 7.2.

First, we shall begin with some generalized results, deriving from the first section

Definition 2.0.3. *Let us consider a n-tuple $x \in (\mathbb{F}_2)^n$. We shall call x s-*distributed *if the number of 1's in x is equal to s. Let $c_l(n, s)$ denote the number of s-distributed n-tuples $x \in (\mathbb{F}_2)^n$, where $\alpha^i(x)$ are s-distributed for all $1 \leq i \leq l$.*

We can construct an analog for the classical problem.

Lemma 2.0.2. *Let us consider a given s-distributed n-tuple $x \in (\mathbb{F}_2)^n$. We have that $\alpha(x)$ is s-distributed, if and only if x is of the form*

$$\delta_0 = \left(a_0, b_0, \ldots, a_{s/2-1}, b_{s/2-1}\right).$$

Proof. Suppose that x is of the form $\delta_0 = (a_0, b_0, \ldots, a_{k-1}, b_{k-1})$. Let us consider a maximal group of r consecutive equal elements. We have that after the transformation it maps into a group of $r-1$ 0's and a group of single 1 (Note that r can be equal to 1). Hence,

$$\mu(\alpha(x)) = \sum_{i=0}^{k-1} (a_i + b_i) - 2k = n - 2k. \tag{2.1}$$

We have that $\alpha(x)$ is s-distributed which is equivalent to $\mu(\alpha(x)) = n - s$. The positioning of the 1's is irrelevant. Now, using (2.1) we have that $k = s/4$, which completes the proof. □

Proposition 2.0.3. *The following is true:*

$$c_1(n,s)=\binom{s-1}{s/2}\left(\binom{n-s}{s/2}+2\binom{n-s-1}{s/2-1}\right)$$

Proof. Let us consider the number of n-tuples $c_{1,1}(n,s)$ of the form $\delta_1=(a_0,b_0,\ldots,a_{k-1},b_{k-1})$, $c_{1,2,1}(n,s)$ of the form $\delta_2=(a_0,b_0,\ldots,a_{k-1},b_{k-1},a_k)$ and $c_{1,2,2}(n,s)$ of the form $\delta_2=(b_0,a_0,\ldots,a_{k-1},a_{k-1},b_k)$ with the required properties. Hence, due to symmetry we have

$$c_1(n,s)=2c_{1,1}(n,s)+c_{1,2,1}(n,s)+c_{1,2,2}(n,s).$$

First we shall derive a formula for $c_{1,1}(n,s)$. From Lemma 4 we have that $k=s/2$. Hence, the necessary and sufficient conditions are

$$\begin{cases}\sum_{i=0}^{s/2-1}a_i=s,\\ \sum_{i=0}^{s/2-1}b_i=n-s.\end{cases}.$$

We should note that $a_i\geq 1$ and $b_i\geq 1$. Now, using Lemma (1.1.2), we have that the number of solutions of $\sum_{i=0}^{s/2-1}a_i=s$ is $\binom{s-1}{s/2-1}$. The number of solutions of $\sum_{i=0}^{s/2-1}b_i=n-s$ is $\binom{n-s-1}{s/2}$. Thus,

$$c_{1,1}(n,s)=\binom{s-1}{s/2-1}\binom{n-s-1}{s/2}. \tag{2.2}$$

Now, we shall consider $c_{1,2,1}(n,s)$. We have that the number of the a-elements is $s/2+1$ and the number of the b-elements is $s/2$. Hence, we face the following conditions

$$\begin{cases}\sum_{i=0}^{s/2}a_i=s,\\ \sum_{i=0}^{s/2-1}b_i=n-s.\end{cases}.$$

Thus, from Lemma (1.1.2) we have that

$$c_{1,2,1}(n,s)=\binom{s-1}{s/2}\binom{n-s-1}{s/2-1}. \tag{2.3}$$

Analogically, the necessary conditions for $c_{1,2,2}(n,s)$ are

$$\begin{cases}\sum_{i=0}^{s/2}b_i=n-s,\\ \sum_{i=0}^{s/2-1}a_i=s.\end{cases},$$

which leads us to

$$c_{1,2,2}(n,s)=\binom{s-1}{s/2-1}\binom{n-s-1}{s/2}. \tag{2.4}$$

Now, from 2.2, 2.3 and 2.4 the statement follows. □

Proposition 2.0.4. *The following is true:*

$$c(s)_2=\sum_{i=1}^{\min\{s/2-1,n-3/2s\}}\binom{s/2}{i}\binom{s/2-1}{i}(3X_i+Y_i)$$

where

$$X_i=\binom{s/2}{i}\binom{n-3/2s-1}{i-1}$$

and

$$Y_i = \binom{s/2-1}{i-1}\binom{n-3/2s}{i}.$$

Proof. We have that x, $\alpha(x)$ and $\alpha^2(x)$ are s-distributed. From Lemma (2.0.2) we have that the necessary and sufficient condition for $\alpha^2(x)$ to be s-distributed is $k = s/2$ if $\alpha(x)$ is of the form δ_i, for $i = 1, 2$. Hence, after the transformation of x, we should obtain a n-tuple with $k = s/2$. Lemma (2.0.2) states that if x is of the form
δ_i $i = 1, 2$, then $k = s/2$. Now, let us consider x as the form $\delta_0 = (a_0, b_0, \ldots, a_{k-1}, b_{k-1})$. Every group with more than one elements, after the transformation, is shorter, but remains. If we consider a group of length one, then after the transformation it is deleted. We should note that between two groups of length greater than one, we obtain a new group. Now, let us have exactly r groups of length one in $x = \delta_0$. From the above considerations we have the following equation:

$$2\frac{s}{2} - 2r = s$$

Thus, $r = s$ and it also is a sufficient condition. Let us denote with $c_{2,1}(n, s)$ the number of n-tuples with the properties we are looking for of the form δ_1, with $c_{2,2,1}(n, s)$ - the number of these of the form $\left(a_0, b_0, \ldots, a_{s/2-1}, b_{s/2-1}, a_{s/2}\right)$ and with $c_{2,2,2}(n, s)$ - the number of the n-tuples of the form
$\left(b_0, a_0, \ldots, b_{s/2-1}, a_{s/2-1}, b_{s/2}\right)$. First we shall calculate $c_{2,1}(n, s)$. We have the following conditions:

$$\begin{cases} \sum_{i=0}^{s/2-1} a_i = s, \\ \sum_{i=0}^{s/2-1} b_i = n - s, \\ s/2 \text{ distributed elements of length one.} \end{cases}$$

Let us have r elements of length one among all a_i, $\forall$ i. Then, we have $s/2 - r$ elements, distributed among the b-elements. There are $\binom{s/2-1}{r}\binom{s/2}{r}$ ways of placing the 'ones'. Now, using Lemma (1.1.2), we have that

$$c_{2,1}(n, s) = \sum_{r=1}^{\min\{s/2-1, n-3/2s\}} \binom{s/2}{r}^2 \binom{s/2-1}{r}\binom{n-3/2s-1}{r-1}.$$

We continue with deriving the formula for $c_{2,2,1}(n, s)$. We have the following conditions:

$$\begin{cases} \sum_{i=0}^{s/2} a_i = s, \\ \sum_{i=0}^{s/2-1} b_i = n - s, \\ s/2 \text{ distributed elements of length one, except } a_0 \text{ and } a_{s/2}. \end{cases}$$

Once again we have r elements with length one among the a-elements. Hence, there are $\binom{s/2-1}{r}\binom{s/2}{s/2-r} = \binom{s/2-1}{r}\binom{s/2}{r}$ ways of placing the 'ones'. Now, using Lemma 2 we have that

$$c_{2,2,1}(n, s) = \sum_{r=1}^{\min\{s/2-1, n-3/2s\}} \binom{s/2}{r}^2 \binom{s/2-1}{r}\binom{n-3/2s-1}{r-1}.$$

We shall consider $c_{2,2,2}(n,s)$. There are the following conditions:

$$\begin{cases} \sum_{i=0}^{s/2-1} a_i = s, \\ \sum_{i=0}^{s/2} b_i = n-s, \\ s/2 \text{ distributed elements of length one, except } b_0 \text{ and } b_{s/2}. \end{cases}$$

Proceeding analogically to the case of $c_{2,2,1}(n,s)$ we obtain the following formula:

$$c_{2,2,2}(n,s) = \sum_{r=1}^{\min\{s/2-1, n-3/2s\}} \binom{s/2}{r}\binom{s/2-1}{r}\binom{s/2-1}{r-1}\binom{n-3/2s}{r}.$$

Now, due to symmetry we have that $c_2(n,s) = 2c_{2,1}(n,s) + c_{2,2,1}(n,s) + c_{2,2,2}(n,s)$, to complete the proof. □

Now, we will investigate another type of generalization, which where we shall use the following definitions:

Definition 2.0.4. *Let $\mu_i(x,r)$ denote the number of elements, equal to i, where $0 \le i < r$ in a given n-tuple x in $(\mathbb{F}_r)^n$. We shall call x* r-balanced *if $\mu_i(x,r) = n/r$, $\forall 0 \le i < r$. If x is r-balanced and $\alpha^i(x)$ is r-balanced for every $i \in \mathbb{N}$ we will call x* super-r-balanced.

From the definition we have that if a n-tuple is r-balanced, then n is divisible by r. We continue with results on these n-tuples.

Definition 2.0.5. *Let us denote with $c(n,3)$ the number of cyclic n-tuples $x \in (\mathbb{F}_3)^n$ (i.e. starting and finishing with different elements) with the property that they itself are 3-balanced and the image $\alpha(x)$ is 3-balanced as well.*

Proposition 2.0.5. *We have that*

$$c(n,3) = 2\sum_{j=1}^{n/3} Q_j \binom{n/3-1}{j-1}^3,$$

where

$$Q_j = \sum_{i=0}^{\lceil j/2 \rceil} \binom{j-1}{i}\left(\binom{j-1}{i}\left(\binom{2j+1-2i}{j+1} - 3\binom{2j-1-2i}{j+1}\right) + \binom{j-1}{i+1}\left(\binom{2j-2i}{j+1} - 3\binom{2j-2i-2}{j+1}\right)\right).$$

Proof. Similarly to the classical problem, we are going to consider the form

$$\Delta = (a_0, a_1, \ldots, a_{k-1}, a_k),$$

meaning that if we have fixed a given n-tuple on a circle, starting from a point in a given direction we have in order respectively a_0 consecutive equal elements, then a_1 consecutive equal elements, but different to these, counted in a_0, etc. In other words a_i denotes the number of consecutive equal elements in the i^{th} cluster, in the order. We also should note that the elements, counted in a_i are different then the elements, counted in its neighboring elements in Δ.

For example, if we take $x = (0,0,1,2,2,2,0,1,1,0,2,1)$, then the form Δ of x is $(2,1,3,1,2,1,1,1)$.

We should note that $0 = \{0+0, 1+2\}$, $1 = \{2+2, 0+1\}$, $2 = \{1+1, 0+2\}$ in the ring $\mathbb{F}_3$. Now, let us suppose that we have k groups in the from Δ of a given x that we should count in $c(n,3)$. Let us have k_0 groups of consecutive 0's, i.e. k_1 groups that count 0's in Δ. Analogously we consider k_1 groups of consecutive 1's and k_2 groups of consecutive 2's.

Now, if we consider the groups, counting consecutive 0's as $y_1, y_2, \dots, y_{k_0}$, we have that after the transformation a group of y_1 zeroes goes to a group of $y_1 - 1$ zeroes. On the other hand we have that we want to keep the number of the zeroes. Hence, the number of consecutive groups, one of them which counts 1's and the other counts 2's is exactly k_0, because $0 = 1+2$ in the given ring. Now, let us order the groups of consecutive 0's. We know that there are exactly k_0. Hence, there are k_1 sequences of group, counting 1's and 2's between pairs of groups that count 0's (because we are considering cyclic sequences). Let z_i denote the number of groups in a given sequence of the upper-considered. The sum of the z's is therefore $k - k_0$. Hence, from the above considerations

$$\sum_{i=1}^{k_0}(z_i - 1) = k_0 \Leftrightarrow k - k_0 - k_0 = k_0 \Leftrightarrow k = 3k_0.$$

Now, if we consider the groups, counting consecutive 1's as $y_1, y_2, \dots, y_{k_1}$, we have that after the transformation a group of y_1 ones goes to a group of $y_1 - 1$ 2's. On the other hand we have that we want to keep the number of the 2's. Hence, the number of consecutive groups, one of them which counts 0's and the other counts 2's is exactly k_1, because $2 = 0+2$ in the given ring. Now, let us order the groups of consecutive 1's. We know that there are exactly k_1. Hence, there are k_0 sequences of group, counting 0's and 2's between pairs of groups that count 1's (because we are considering cyclic sequences). Let z_i denote the number of groups in a given sequence of the upper-considered. The sum of the z's is therefore $k - k_1$. Hence, from the above considerations

$$\sum_{i=1}^{k_1}(z_i - 1) = k_1 \Leftrightarrow k - k_1 - k_1 = k_1 \Leftrightarrow k = 3k_1.$$

Now, if we consider the groups, counting consecutive 2's as $y_1, y_2, \dots, y_{k_1}$, we have that after the transformation a group of y_1 ones goes to a group of $y_1 - 1$ 1's. On the other hand we have that we want to keep the number of the 1's. Hence, the number of consecutive groups, one of them which counts 0's and the other counts 1's is exactly k_2, because $1 = 0+1$ in the given ring. Now, let us order the groups of consecutive 1's. We know that there are exactly k_1. Hence, there are k_2 sequences of group, counting 0's and 1's between pairs of groups that count 2's (because we are considering cyclic sequences). Let z_i denote the number of groups in a given sequence of the upper-considered. The sum of the z's is therefore $k - k_2$. Hence, from the above considerations

$$\sum_{i=1}^{k_2}(z_i - 1) = k_2 \Leftrightarrow k - k_2 - k_2 = k_2 \Leftrightarrow k = 3k_2.$$

Therefore, we have that k is divisible by three and $k_0 = k_1 = k_2$. Because we were counting the number of 0's, 1's and 2's we have that the necessary and sufficient conditions for a cyclic n-tuple x to be 3-balanced and its image $\alpha(x)$ to be 3-balanced are

$$\begin{cases} \sum_{i=0}^{k/3-1} M_i = n/3, \\ \sum_{i=0}^{k/3-1} N_i = n/3, \\ \sum_{i=0}^{k/3-1} P_i = n/3. \end{cases}$$

where M_i, N_i, P_i denote respectively the elements in Δ that count the number of consecutive 0's 1's and 2's and we have k groups in Δ. Now suppose that we have

$3j$ group and a fixed Δ. Furthermore,

$$\sum_{i=0}^{\lceil j/2\rceil}\binom{j-1}{i}\left(\binom{j-1}{i}\left(\binom{2j+1-2i}{j+1}-3\binom{2j-1-2i}{j+1}\right)+\binom{j-1}{i+1}\left(\binom{2j-2i}{j+1}-3\binom{2j-2i-2}{j+1}\right)\right)$$

is the number of Δ's (See [2] and further generalization of this problem in [3], suggested by the author of [1]), because actually the problem is finding the number of linear arrangements of $k/3$ blue, $k/3$ red and $k/3$ green items such that there are no adjacent items of the same color (first and last elements considered as adjacent). Considering the sum $\sum_{i=0}^{k/3-1} M_i = n/3$, where $M_i \geq 1$ and using Lemma (1.1.2) we have $\binom{n/3-1}{j-1}$ solutions and respectively the number of distributions for a fixed Δ is

$$\binom{n/3-1}{j-1}^3.$$

Now, parameterizing the number of groups with the upper considerations completes the proof. □

We present answers to the questions of existence of super-r-balanced n-tuples.

Proposition 2.0.6. *Super-r-balanced n-tuples exist if r is odd.*

Proof. Let us take n divisible by r and consider the sequence $x = (x_0, x_1, \ldots x_{n-1})$, where $x_i = i \mod r$. We are going to prove that this n-tuple is super-r-balanced. First we shall consider only:

$$a = (a_0, a_1, \ldots, a_{r-1}),$$

where $a_i = i$. Let us consider $\alpha^l(a) = (a_0(l), a_1(l), \ldots, a_{r-1}(l))$. We shall prove by induction that

$$a_i(l) = \sum_{j=0}^{l} a_{i+j}\binom{l}{j}$$

Indeed, for $l = 1$ this statement is correct. Now, suppose that the statement is correct for $l \geq 2$. By definition of α we have that

$$\begin{aligned} a_i(l+1) &= a_i(l) + a_{i+1}(l) \\ &= \sum_{j=0}^{l}\left(a_{i+j}\binom{l}{j} + a_{i+j+1}\binom{l}{j}\right) \\ &= a_i + \sum_{j=1}^{l-1} a_{i+j}\left(\binom{l}{j} + \binom{l}{j-1}\right) + a_{l+i+1} \end{aligned}$$

and using that $\binom{l}{j} + \binom{l}{j-1} = \binom{l+1}{j}$ we complete the induction.

Now, let us suppose that in $\alpha^l(a)$ we have two elements on position s and r which are equal. Hence, we have that

$$a_s(l) = a_r(l) \Leftrightarrow \sum_{j=0}^{l} a_{s+j}\binom{l}{j} = \sum_{i=0}^{l} a_{r+i}\binom{l}{i}$$

Now we use the property of a, that is we have consecutive numbers. Let $a_s = S$ and $a_r = R$. Hence,

$$S+(S+1)\binom{l}{1}+(S+2)\binom{l}{2}+\cdots+(S+l) = R+(R+1)\binom{l}{1}+(R+2)\binom{l}{2}+\cdots+(R+l).$$

From the last equation we have that if $a_s(l) = a_r(l)$ and $s \neq r$, then $R = S$, which is a contradiction with a. So, we know have that every image $\alpha^i(a)$ is a permutation of the numbers in a, i.e. every image of a is r-balanced. Now we should note that we have n/r consecutive sequences a, that are independent, i.e. every subsequence of type a is r-balanced for every transformation. Hence, the whole sequence x is r-balanced for every transformation, which completes the proof. □

It is logical to ask ourselves about the number of super-r-balanced n-tuples. It turns out that these sequences, that we have considered in the proof are not the only super-r-balanced one. In fact the sequence $(0, \dots, 0, 1, \dots, 1, 2, \dots, 2, \dots, (r-1), \dots (r-1)$, where we have n/r elements of every $0 \leq i < r$ are super-r-balanced as well. One thing we can see is that the number of super-r-balanced n-tuples is bigger than n. We can obtain that by considering the polynomial $P_x(t)$ and again as in Lemma (1.2.1) to obtain an equation which does not have a solution for $l < n$. That means that the sequences we have considered are transforming to their-selves after more than $n-1$ transformation, which on the other hand means that we have more than $n-1$ super-r-balanced n-tuples. Of course every of these observations is true for r - odd.

Conjecture 2.0.2. *Super-r-balanced n-tuples do not exist for r an even number.*

Remark 2.0.1. *Another interesting way of considering this topic is the following. Let us take all balanced n-tuples ana create a graph, with vertexes, these balanced n-tuples and an oriented edge, if we can go from a vertex to another with one α transformation. We can study the properties of this graph. The first one, which we have proven is that the obtained graph is actually a tree (i.e. there are no cycles), because there are not super-balanced n-tuples. Another interesting result is proving that the matrix of adjacencies is actually nilpotent.*

Another type of generalization may be considering transformation that take the sum of more then 2 consecutive elements.

We should also note that we can consider the set of all balanced n-tuples as a vector space (commonly used in Code Theory). For example, we can easily find the dimension of this vector space. Of course it is n.

References

[1] T. Andreescu, Z. Feng " Path to Combinatorics for Undergraduates. Counting Strategies.". (69-91)

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[3] L. Q. Eifler, K. B. Reid Jr., D. P. Roselle "Sequences with adjacent elements unequal". Aequationes Mathematicae (1971), 6 (2-3), (256-262)

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