PROBLEM 5: STABLE POLYGONS

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Abstract. In the following paper we investigate the properties of the stable sets, defined for a convex n-gon. During the process of research, we use the powerful De Moivre's Theorem. We present combinatorial and geometrical observations and considerations to obtain the rest of the results. In total, we find the number of the stable sets for part of the given cases.

Theorem 1 (De Moivre)**.** *If $z = r(\cos(\emptyset) + i\sin(\emptyset))$ and n is a positive integer, then $z^n = [r(\cos(\emptyset) + i\sin(\emptyset))]^n = r^n(\cos(n\emptyset) + i\sin(n\emptyset))$.*

Lemma 1. *If P is a prime number and a_0, a_1,…,$a_{p-1} \in \mathbb{Q}$ satisfy the relation $a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_{p-1}\varepsilon^{p-1} = 0$, where $\varepsilon = \cos(\frac{2\Pi}{p}) + i\sin(\frac{2\Pi}{p})$ then $a_0 = a_1 = \cdots = a_{p-1}$.*

Proof. It is enough to observe, that the polynomials $a_0+a_1x+a_2x^2+\cdots+a_{p-1}x^{p-1}$ and $1+x+x^2+\cdots+x^{p-1}$ cannot be relatively prime, because they share a common root and since $1 + x + x^2 + \cdots + x^{p-1}$ is irreducible over $\mathbb{Q}$, $1 + x + x^2 + \cdots + x^{p-1}$ must divide $a_0 + a_1x + a_2x^2 + \cdots + a_{p-1}x^{p-1}$, which can only happen if $a_0 = a_1 = \cdots = a_{p-1}$. □

Problem 1. *When n is prime, find the number of stable subsets $A \subset P_n$ and describe them.*

Solution: Let us set up coordinate system with center O and $A_0 \subset Ox$. O is the center of gravity, hence the sum of all participating roots of unity is 0. From Moivre's Theorem the j^{th} root of unity equals ε^j. Now using the lemma, we assume $a_0 = a_1 = \cdots = a_{p-1}$. In our case a_j is equal to 0 or 1, hence, the only polygon is the p-polygon. The answer is 1.

Problem 2. *The same problem when n is the product of two distinct prime numbers.*

Solution Let us numerate the vertices clockwise from A_1 to A_n (as the polygon is right there is no significance where the vertex which is numerated one is). Let us denote by *'defining vertex'* the vertex of a polygon with the smallest number.

Let $n = pq$. Let $p < q$. We have got 2 cases

Case 1. If $p > 2$. Let a k-regular-polygon have the same center of gravity as our polygon. As it is regular that means that every k^{th} point needs to participate in it. Those points divide the circle in which our polygon is inscribed into arcs whose co-responding angle equals $\frac{360}{k}$. We know that our polygon divides the same circle into $\frac{360}{pq}$ arcs, therefore each of the former arcs consist of $\frac{\frac{360}{k}}{\frac{360}{k}} = \frac{pq}{k}$ of the latter arcs. As we need this number to be an integer $k|pq \Rightarrow k|p$ or $k|q$ (if $k|pq$ but $k \nmid p$ and $k \nmid q \Rightarrow k = pq$). As $k > 2$ and p and q are prime, $k = p$ or $k = q$. If $k = p$ there are exactly q different polygon (as there are q numbers between 1 and

q and we construct the polygons by taking such a point and every p^{th} point after that. For the same reason we have p different q-polygons. If we take $z \in \mathbb{N}$ different p-polygons and combine their vertices we generate a new polygon that has the same center of gravity ($\sum_{d=1}^{q}(O\vec{A_{dp+i}} = 0$ where i is the defining vertex of the polygon, therefore by adding those sums equaling zero together they will still equal zero). As no vertex is common for 2 polygons there is no vector that has been counted twice. So the number of different zp-polygons is $\binom{q}{z} \Rightarrow$ the sum of all p-polygons and their multiplicative is $\sum_{d=1}^{q-1}\binom{q}{d} = 2^q - \binom{q}{0} - \binom{q}{q} = 2^q - 2$. The exact same goes for q-polygons. We will now prove that we cannot combine a p-polygon with a q-polygon: Let us take a random p-polygon. Let its defining vertex be A_l. Its other vertices will be as follow: A_{q+l},A_{2q+l},...,$A_{(p-1)q+l}$. As $LCD(p, q) = 1$ we know that there are q different residues modulo q for the sequence 0, p, $2p$,...,qp $\Rightarrow$ one of them is l ($l < q$), therefore whichever p-polygon we take it has exactly one common vertex with every other q-polygon (in this case we took the q-polygon with defining vertex A_0 but the only difference in the others is the residue and as we have every residue in our sequence that is not a problem). As the two polygons have a common vertex their sum is $2\sum_{d=1}^{q}(O\vec{A_{dp+i}} - A_p = 0 - A_p \neq 0$, where A_p is the common vertex. Therefore we cannot combine a p-polygon with a q one. So the total number of polygons with the same center of gravity as the given one is $2^q + 2^p - 2 - 2$. We must also add the given polygon so the answer is $2^p + 2^q - 3$.

Case 2. $p = 2$. This case is analogical to the previous one with the one exception that there are no p-polygons since there is no 2-polygon so we must not add $\binom{q}{1}$. So the total number is $2^q - q + 1$.

Problem 3. *The same problem when n is a power of a prime number.*

Solution: Once again, we shall consider two cases:

Case 1. If $p \neq 2$. We can regard n as $p^{t-1}p$, where p is prime. As such we use the method of proving from the second problem and reach the answer. As there are p^{t-1} p-polygons the total number of polygons is $2^{p^{t-1}} - 1$.

Case 2. $p = 2$. This case is analogical to the previous one with the one exception that there are no p-polygons since there is no 2-polygon so we must not add $\binom{2^{t-1}}{1}$. So the total number is $2^{2^{t-1}} - 2^{t-1} - 1$.

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