PROBLEM 4: ISOSCELES TRIANGLES

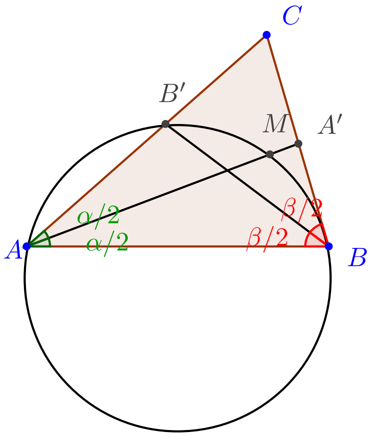
TEAM BULGARIA

Abstract. In the following paper we examine the properties of medians, symmedians, ex-medians and ex-symmedians of a isosceles triangle. We use an apparatus of synthetic geometry to answer the questions in the problem. In the places, where we could not answer the question fully, we present observations that may solve the general problem. We also suggest a further research of the given topic.

If $AB = AC$, then every element constructed from A is congruent to the corresponding from B, due to symmetry, through the perpendicular bisector of AB. By mentioning this we have proven one side of every problem.

Problem (1)**.** *Prove that two internal bisectors of a triangle are equal, if and only if the triangle is isosceles.*

Solution: We are given the triangle ABC, in which $AA^{'}$ and $BB^{'}$ are its internal bisectors ($A^{'} \in BC$, $B^{'} \in AC$). Let us assume that the triangle is not isosceles and that $BC < AC$. Let M be a point from $AA^{'}$, such that $\angle B^{'}BM = \frac{\alpha}{2}$. Now, if we look at the quadrilateral $BMB^{'}A$ we will see that the four points that are constructing this quadrilateral are lying on circle as BM is being overlooked by the same angles from A and B. Now, all we have to mention is that in a circle against the smaller inscribed angles lie shorter chords. From this fact and since $\alpha < \frac{\alpha}{2} + \frac{\beta}{2} = \angle ABM$, we have $BB^{'} < AM < AA^{'}$, which is a contradiction with the fact that in our triangle $AA^{'} = BB^{'}$. So we proved that the triangle must be isosceles if $AA^{'} = BB^{'}$.



Problem (5)**.** *Check that the internal bisectors and the symmedians are respectively internal* $1-lines$ *and* $2-lines$ *of the triangle. Also, the external bisectors and the exsymmedians are respectively external* $1-lines$ *and* $2-lines$ *of the triangle.*

Solution:

1. Let $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$, $\angle CAA_2 = x$ (where AA_2 is the symmedian and $AA_2^{'}$ is the exsymmedian). AA_1 is the internal bisector of $\angle BAC$. From bisector properties we have: $\frac{CA_1}{A_1B} = \frac{CA}{AB} = \frac{b}{c}$. Hence, the internal bisector is $1-line$.

2. $AA_1^{'}$ is the external bisector of $\angle BAC$. From bisector properties we have: $\frac{CA_1^{'}}{A_1^{'}B} = \frac{CA}{AB} = \frac{b}{c}$ and, hence, the external bisector is $1-line$ as well.

3. Now using Sine theorem for $\triangle ACA_2$ we have $\frac{CA_2}{\sin x} = \frac{AC}{\sin\angle AA_2C}$ and from the same theorem for $\triangle ABA_2$ we have : $\frac{BA_2}{\sin(\alpha - x)} = \frac{AB}{\sin\angle AA_2B} = \frac{AB}{\sin\angle AA_2C}$. From this 2 equalities we have: $\frac{BA_2}{CA_2} = \frac{\frac{AB\times\sin(\alpha-x)}{\sin\angle AA_2C}}{\frac{AC\times\sin x}{\sin\angle AA_2C}} = \frac{AB\times\sin(\alpha-x)}{AC\times\sin x} = \frac{c\times\sin(\alpha-x)}{b\times\sin x}$.

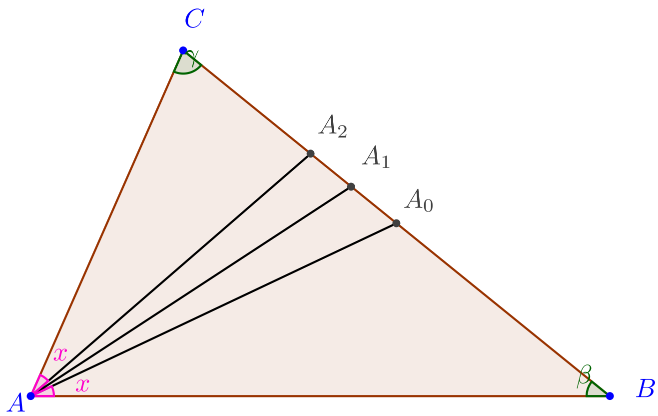
As AA_2 is symmedian we know that $\angle A_2AA_1 = \angle A_0AA_1$ (where AA_0 is the median from A in the $\triangle ABC$)and as AA_1 is internal bisector: $\angle BAA_1 = \angle CAA_1 \Rightarrow \angle BAA_0 + \angle A_0AA_1 = \angle CAA_2 + \angle A_2AA_1 \Rightarrow \angle BAA_0 = \angle CAA_2 = x$.

Using Sine theorem for $\triangle AA_0B$: $\frac{A_0B}{\sin x} = \frac{AA_0}{\sin\beta}$.

Now using the same theorem for $\triangle AA_0C$: $\frac{A_0C}{\sin(\alpha-x)} = \frac{AA_0}{\sin\gamma}$.

From this two equalities: $\frac{\sin x}{\sin(\alpha-x)} = \frac{\frac{A_0B.\sin\beta}{AA_0}}{\frac{A_0C.\sin\gamma}{AA_0}} = \frac{A_0B.\sin\beta}{A_0C.\sin\gamma}$.

We know that AA_0 is a median and therefore we know that $A_0B = A_0C$. Moreover $\frac{\sin x}{\sin(\alpha-x)} = \frac{\sin\beta}{\sin\gamma} \Rightarrow \frac{BA_2}{CA_2} = \frac{c\sin(\alpha-x)}{b\sin x} = \frac{c\sin\gamma}{b\sin\beta}$. Now from Sine theorem for $\triangle ABC$ we have $\frac{\sin\gamma}{\sin\beta} = \frac{c}{b}$. Therefore $\frac{BA_2}{CA_2} = \frac{c^2}{b^2}$ and so the symmedian is internal $2-line$.



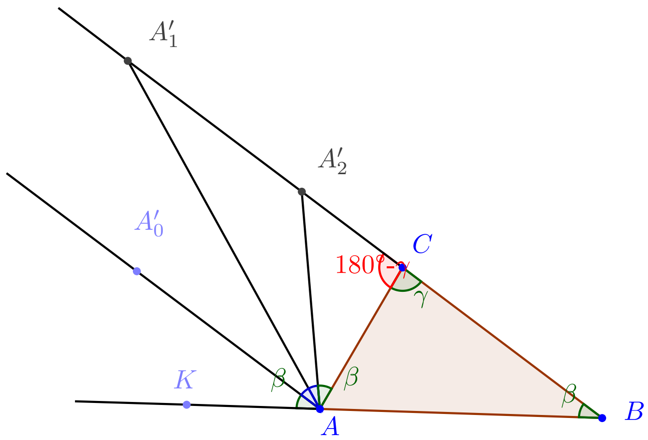
4. Let $K \in AB$ and A is between K and B. $AA_0^{'}$ is the ex-median and therefore $AA_0^{'} \parallel BC \Rightarrow \angle KAA_0^{'} = \angle ABC = \beta$. As $AA_1^{'}$ is external bisector $\angle KAA_1^{'} = \angle CAA_1^{'} \Rightarrow \angle KAA_0^{'} + \angle A_0^{'}AA_1^{'} = \angle A_1^{'}AA_2^{'} + \angle A_2^{'}AC$.

As $AA_2^{'}$ is the ex-symmedian : $\angle A_0^{'}AA_1^{'} = \angle A_1^{'}AA_2^{'} \Rightarrow \angle KAA_0^{'} = \angle A_2^{'}AC = \beta$

Now according to Sine theorem for $\triangle AA_2'C$: $\frac{A_2'C}{\sin\beta} = \frac{AA_2'}{\sin\gamma}$.

And again from the same theorem for $\triangle AA_2'B$: $\frac{A_2'B}{\sin(\alpha+\beta)} = \frac{AA_2'}{\sin\beta}$

From all above we have: $\frac{A_2'C}{A_2'B} = \frac{\frac{\sin\beta . AA_2'}{\sin\gamma}}{\frac{\sin(\alpha+\beta).AA_2'}{\sin\beta}} = \frac{\sin^2\beta}{\sin\gamma.\sin(\alpha+\beta)} = \frac{\sin^2\beta}{\sin\gamma.\sin(180-\gamma)} = \frac{\sin^2\beta}{\sin^2\gamma} = \frac{b^2}{c^2}$. This statement proves that the ex-median is external $2-line$.



Problem (2)**.** *The symmedian through a given vertex of a triangle is constructed by reflecting the median about the internal angle bisectors at the same vertex. Prove that two symmedians of a triangle are equal if and only if the triangle is isosceles.*

Solution: We construct a triangle ABC in which AA_0 and BB_0 are symmedians. From **Problem 5.** we know,that the symmedians are $2-lines$, hence $\frac{BA_0}{A_0C} = \frac{c^2}{b^2}$ and $\frac{AB_0}{B_0C} = \frac{c^2}{a^2}$. From Cosine Theorem $(AA_0)^2 = b^2 + (\frac{b^2a}{b^2+c^2})^2 - 2b\frac{b^2a}{b^2+c^2}\cos(\gamma)$ and $(BB_0)^2 = a^2 + (\frac{a^2b}{a^2+c^2})^2 - 2a\frac{a^2b}{a^2+c^2}\cos(\gamma)$. From Cosine Theorem $\cos(\gamma) = \frac{b^2+a^2-c^2}{2ab}$, hence,

$(AA_0)^2 = b^2 + (\frac{b^2a}{b^2+c^2})^2 - 2b\frac{b^2a}{b^2+c^2}\frac{b^2+a^2-c^2}{2ab}$

$(AA_0)^2 = \frac{2b^4c^4+2b^4c^2-a^2b^2c^2}{(b^2+c^2)^2}$

$(BB_0)^2 = a^2 + (\frac{a^2b}{a^2+c^2})^2 - 2a\frac{a^2b}{a^2+c^2}\frac{a^2+b^2-c^2}{2ab}$

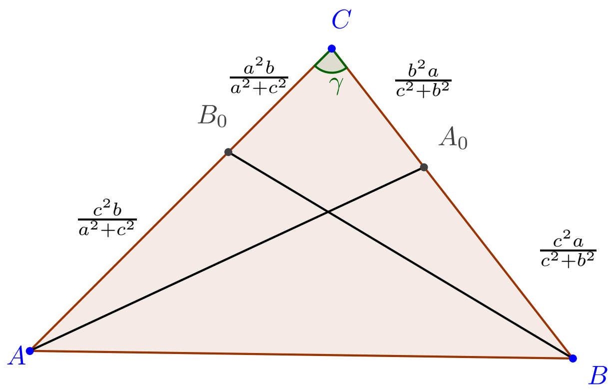
$(BB_0)^2 = \frac{2a^4c^4+2a^4c^2-a^2b^2c^2}{(a^2+c^2)^2}$

We know,that $AA_0 = BB_0 \Rightarrow (AA_0)^2 = (BB_0)^2 \Rightarrow$

$\frac{2b^4c^4+2b^4c^2-a^2b^2c^2}{(b^2+c^2)^2} = \frac{2a^4c^4+2a^4c^2-a^2b^2c^2}{(a^2+c^2)^2} \Rightarrow$

$(2b^4c^4 + 2b^4c^2 - a^2b^2c^2)(a^2 + c^2)^2 = (2a^4c^4 + 2a^4c^2 - a^2b^2c^2)(b^2 + c^2)^2$

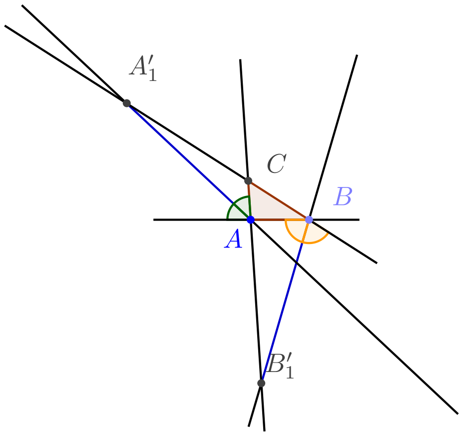
$T = 4a^2b^2c^2(a^2 - b^2) + 2c^6(a^2 - b^2) + 2c^4(a^4 - b^4) + a^2b^2(a^4 - b^4) = 0$ If $a > b$ $T > 0$ and if $a < b$ $T < 0 \Rightarrow a = b$ $BC = AC$ and therefore the triangle is isosceles.



Problem (3)**.** *Is it true that two external angle bisectors of a triangle are equal if and only if the triangle is isosceles?*

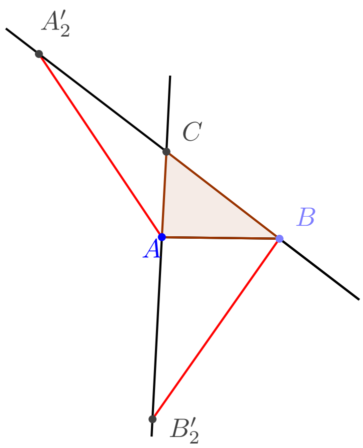
Solution: Let us assume that $a > b$. From bisector $\frac{A_1'C}{A_1'B} = \frac{AC}{AB} = \frac{b}{c} \Rightarrow \frac{A_1'C}{A'C+a} = \frac{b}{c} \Rightarrow A_1'C = \frac{ab}{c-b}$.

From Cosine Theorem for $\triangle ACA_1' \Rightarrow$
$AA_1'^2 = AC^2 + A_1'C^2 - 2AC.A_1'C\cos(180-\gamma) = b^2 + \frac{a^2b^2}{(c-b)^2} + 2b\frac{ab}{c-b}\cdot\frac{a^2+b^2-c^2}{2ab} == b^2 + \frac{a^2b^2}{(c-b)^2} + b\frac{a^2+b^2-c^2}{c-b} = b(b+\frac{a^2b}{(c-b)^2} + \frac{a^2+b^2-c^2}{c-b}) = b(bc^2 - \frac{2b^2c+b^3+a^2b+a^2c+b^2c-c^3-a^2b-b^3+bc^2}{(c-b)^2}) = b\frac{2bc^2-b^2c+a^2c-c^3}{(c-b)^2} = bc\frac{2bc-b^2+a^2-c^2}{(c-b)^2} = bc\frac{a^2-(b-c)^2}{(b-c)^2} = \frac{bc(a-b+c)(a+b-c)}{(b-c)^2}$
Analogically $BB_1'^2 = \frac{ac(b-a+c)(a+b-c)}{(a-c)^2}$
Let $AA_1' = BB_1' \Rightarrow AA_1'^2 = BB_1'^2 \Rightarrow \frac{bc(a-b+c)(a+b-c)}{(b-c)^2} = \frac{ac(b-a+c)(a+b-c)}{(a-c)^2}$
$a+b > c \Rightarrow a+b-c \neq 0; c \neq 0 \Rightarrow \frac{b(a-b+c)}{(b-c)^2} = \frac{a(b-a+c)}{(a-c)^2}$. Therefore $(a-c)^2b(a-b+c)-(b-c)^2a(b-a+c) = 0$. Furthermore $(a-b)(c^3-c^2b-c^2a-a^2c-ac^2+3abc) = 0$. As $a-b \neq 0$ $(a > b)$, $c^3 - c^2b - c^2a - a^2c - ac^2 + 3abc = 0$. Now we see that if $c = 3, b = 2$ and $a = \frac{5+\sqrt{97}}{4}$, we have a triangle with equal bisectors and this triangle is not isosceles.



Problem (4)**.** *An ex-median is a parallel to a side of a triangle through the opposite vertex. The ex-symmedian through a vertex of a triangle is constructed by reflecting the ex-median about the external angle bisectors at the same vertex. Is it true that two ex-symmedians of a triangle are equal if and only if the triangle is isosceles?*

Solution: We assume that the triangle is not isosceles and $a > c > b$. In **Problem 5.** we proved that $\frac{BA_2^{'}}{CA_2^{'}} = \frac{c^2}{b^2} = \frac{a+A_2^{'}C}{A_2^{'}C}$. Therefore $A_2^{'}C = \frac{ab^2}{c^2-b^2}$. From Sine theorem for $\triangle AA_2'C$: $AA_2^{'} = b^2 + \frac{a^2b^4}{(c^2-b^2)^2} - 2\frac{ab^3}{c^2-b^2}.\cos(180-\gamma) = b^2 + \frac{a^2b^4}{(c^2-b^2)^2} + 2\frac{ab^3}{c^2-b^2}\frac{b^2+a^2-c^2}{2ab} = \frac{(abc)^2}{(c^2-b^2)^2}$. Analogically $BB_2' = \frac{(abc)^2}{(c^2-a^2)^2}$. From the problem's statement we know that $AA_2^{'} = BB_2^{'}$, so $\frac{(abc)^2}{(c^2-b^2)^2} = \frac{(abc)^2}{(c^2-a^2)^2}$. Hence $abc \neq 0 \Rightarrow (c^2-a^2)^2 - (c^2-b^2)^2 = 0$. Furthermore $(b-a)(b+a)(2c^2-a^2-b^2) = 0$ b and a are sides of a triangle, hence $b, a > 0$ and $a \neq b$ from the assumption that the triangle is not isosceles. Therefore $2c^2 - a^2 - b^2 = 0$. If $a = 5, b = 3$ and $c = \sqrt{17}$, we have a triangle with equal ex-symmedians and this triangle is not isosceles.



Problem (6)**.** *Is is true that two internal $n-lines$ of a triangle are equal if and only if the triangle is isosceles?*

Solution: From $n-lines$ $\frac{CA_n}{A_nB} = \frac{b^n}{c^n}$. From Cosine Theorem for AA_nC and $BB_nC \Rightarrow$

$AA_n^2 = b^2 + (\frac{b^n}{b^n+c^n}a)^2 - 2b\frac{b^n}{b^n+c^n}a\cos(\gamma)$

$BB_n^2 = a^2 + (\frac{a^n}{a^n+c^n}b)^2 - 2a\frac{a^n}{a^n+c^n}b\cos(\gamma)$.

From Cosine Theorem $\cos(\gamma) = \frac{b^2+a^2-c^2}{2ab} \Rightarrow$

$AA_n^2 = \frac{b^{n+2}c^n+b^2c^{2n}+b^{2n}c^2-b^nc^na^2+b^nc^{n+2}}{(b^n+c^n)^2}$

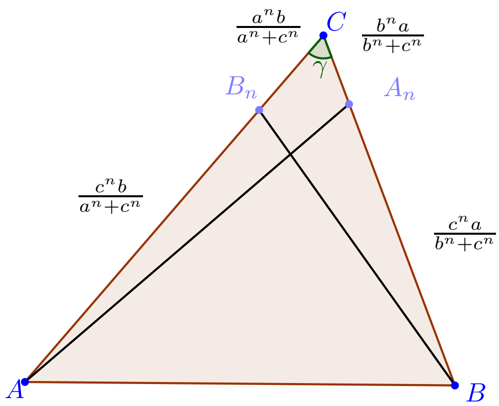
$BB_n^2 = \frac{a^{n+2}c^n+a^2c^{2n}+a^{2n}c^2-a^nc^nb^2+a^nc^{n+2}}{(a^n+c^n)^2}$.

We know, that if the triangle is isosceles, $AA_n = BB_n$, hence we are in case 2, where $AA_n = BB_n \Rightarrow$

$AA_n^2 = BB_n^2 \Rightarrow$

$(a^{n-2} - b^{n-2})(a^{n+2}b^{n+2}c^n + 3a^2b^2c^{3n}) - (a^{n+2} - b^{n+2})(a^nb^nc^n + c^{3n}) - (a^n - b^n)(a^nb^nc^{n+2} + c^{3n+2}) - (a^2 - b^2)(4a^nb^nc^{2n} + c^{4n}) + a^2b^2c^{2n}(a^{2n-2} - b^{2n-2}) - (a^{2n} - b^{2n})c^{2n+2} = 0$

We suspect $a = b$ to be the only solution, although we are unable to prove it.



Problem (7)**.** *Is it true that two external $n-lines$ of a triangle are equal if and only if the triangle is isosceles?*

Solution: From $n-lines$ $\frac{BA_n}{CA_n} = \frac{c^n}{b^n} \Rightarrow CA_nb^n + ab^n = CA_nc^n \Rightarrow CA_n = a\frac{b^n}{c^n-b^n}$ From Cosine Theorem for AA_nC and $BB_nC \Rightarrow AA_n^2 = b^2 + (\frac{b^n}{c^n-b^n}a)^2 + 2b\frac{b^n}{c^n-b^n}a\cos(\gamma)$

$BB_n^2 = a^2 + (\frac{a^n}{c^n-a^n}b)^2 + 2a\frac{a^n}{c^n-a^n}b\cos(\gamma)$.

From Cosine Theorem $\cos(\gamma) = \frac{b^2+a^2-c^2}{2ab} \Rightarrow$

$AA_n^2 = \frac{c^{2n}b^2+c^2b^{2n}-b^{n+2}c^n-b^nc^{n+2}+b^nc^na^2}{(c^n-b^n)^2}$

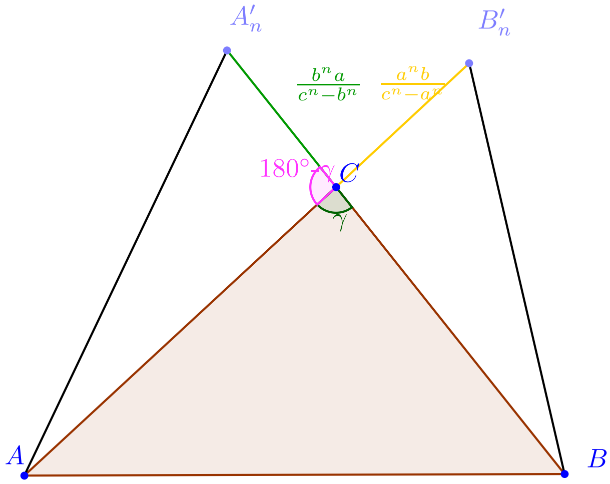
$BB_n^2 = \frac{c^{2n}a^2+c^2a^{2n}-a^{n+2}c^n-a^nc^{n+2}+a^nc^nb^2}{(c^n-a^n)^2}$.

We know, that if the triangle is isosceles, $AA_n = BB_n$, hence we are in case 2, where $AA_n = BB_n \Rightarrow$

$b^2c^{4n} + c^{2n+2}b^{2n} - b^{n+2}c^{3n} - c^{3n+2}b^n + a^2b^nc^{3n} - 2a^nb^2c^{3n} - 2a^nb^{2n}c^{n+2} + 2a^nb^{n+2}c^{2n} - 2a^{n+2}b^nc^{2n} + a^{2n}b^2c^{2n} - a^{2n}b^{n+2}c^n - a^{2n}b^nc^{n+2} + a^{2n+2}b^nc^n - a^2c^{4n} -$

$a^{2n}c^{2n+2} + a^{n+2}c^{3n} + a^n c^{3n+2} - a^n b^2 c^{3n} + 2a^2 b^n c^{2n+2} - 2a^{2n} b^n c^{n+2} - 2a^{n+2} b^n c^{2n} + 2a^n b^{n+2} c^{2n} - a^2 b^{2n} c^{2n} + a^{n+2} b^{2n} c^n + a^n b^{2n} c^n + a^n b^{2n} c^{n+2} - a^n b^{2n+2} c^n = 0$

Although we cannot prove it, we expect that $a = b$ is not the only answer.



Problem (8). *Suggest and study additional directions of research.*

It is sufficiently interesting to consider the object n-point, which is defined as the crossing point of the upper-considered n-lines. They intersect in a single point and we can easily prove that by using Cheva's Theorem. The properties of this object represent an interest to the mathematical world (see [1]).

References

[1] Bulgarian journal "Matematika", issue 1, 2008, (10-17)

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