PROBLEM 1: GENERALIZING PERFECTNESS

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Abstract. In the following paper we analyze the properties of different f-perfect functions. During the process of research we use a variety of Theory of Numbers tools. We firmly and comprehensively answer most of the questions in the problem and present our ideas and consideration in the places, where we did not manage to generalize the sub-question.

Problem 1. *Let $\tau(n)$ denote the number of positive divisors of n(including n).*

a) Prove that a natural number $n \geq 1$ is $\tau - perfect$ if and only if n is a square of a prime.

b) Find all f-perfect *numbers $n \geq 1$ for the function $f(n) = \tau(n) - 1$. For as many values of $k \in \mathbb{Z}$ as possible, find all f*-perfect *natural numbers $n \geq 1$ for $f(n) = \tau(n) + k$*

Solution: First of all, we will prove the following theorem:

Theorem 1. *Let the canonical representation of n be $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. Then we have $\tau(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1)$.*

Proof. We must mention that $\tau(n)$ is a multiplicative function. Each divisor of n has a canonical representation $p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$, where $0 \leq b_i \leq a_i$ for every $i = 1, 2, \dots, k$. Each index b_i can be chosen by $a_i + 1$ ways. Therefore, from the rule for multiplying possibilities, we can conclude that the number of the different divisors of n is exactly $(a_1 + 1)(a_2 + 1) \dots (a_k + 1)$. □

a) Let n be the square of p (p is prime number). From **Theorem 1** we have:

$$\tau(n) = \tau(p^2) = (2 + 1) = 3(1) \tag{0.1}$$

Moreover we know that all divisors of n are 1, p, p^2. Therefore we know that $\sum_{d|n} = \tau(1) + \tau(p)$. (since $p^2 \not< n$) From **Theorem 1** we have $\tau(p) = (1 + 1) = 2$ (p is prime.) and $\tau(1) = 1$. So,

$$\sum_{d|n} = \tau(1) + \tau(p) = 1 + 2 = 3. \tag{0.2}$$

From (0.1) and (0.2) we come to the conclusion that if n is a square of a prime, n is $\tau - perfect$.

Now we will prove that if n is $\tau - perfect$, n is the square of a prime number. Let the canonical representation of n be $n = p_1^{a_1} p_2^{a_2} \dots p_z^{a_z}$. If $z > 2$ and at least one $a_i \geq 2$ (where i=1,2,. . . ,z). On the one hand $\sum_{d|n, d \leq n-1} \tau(d) > \tau(\frac{n}{p_i}) + \tau(\frac{n}{p_i^{a_i}})$. On the other hand $\tau(\frac{n}{p_i}) + \tau(\frac{n}{p_i^{a_i}}) = \tau(n)$. This leads us to a contradiction.If $z = 2$ and $a_1, a_2 > 1 \Rightarrow n = p_1^{a_1} p_2^{a_2}$ then $\tau(n)$=$(a_1 + 1)(a_2 + 1)$ and $\sum_{d|n}(\tau(n)) > \tau(\frac{n}{p_1}) + \tau(\frac{n}{p_2}) = (a_1 + 1)(a_2 + 1) + a_1 a_2 - 1 > (a_1 + 1)(a_2 + 1)$. Therefore there is no solution. Hence, either $z < 3$, all $a_i = 1$, or both.

If $n = p^a$: $f(n) = \tau(n) = a + 1 = \frac{a(a+1)}{2} = \sum_{d|n,d\leq n-1} \tau(d)$. Therefore $a + 1 = \frac{a(a+1)}{2} \Rightarrow a = 2$ which means that n is the square of a prime number.

If $n = p_1p_2 \dots p_z$, then $\tau(n) = 2^z$. If $z = 2$, it is obvious that n is not τ-perfect as $\tau(n) = 4$ and $\sum_{d|n,d\leq n-1} \tau(d) = 5$. If $z \geq 3$, $\sum_{d|n,d\leq n-1} \tau(d) > \tau(\frac{n}{p_1}) + \tau(\frac{n}{p_2}) = 2^{z-1} + 2^{z-1} = 2^z$. To conclude, we proved that $\sum_{d|n,d\leq n-1} \tau(d) > 2^z = \tau(n)$ in every other case than $z = 1$, a_1=2.

b) We will start by solving the case, where $f(n) = \tau(n) - 1$. Let n be a power of a prime ($n = p^z$). On the one hand $f(n) = \tau(n) - 1 = z$. On the other hand $\sum_{d|n,d<n-1}(\tau(d) - 1) = 0 + 1 + 2 + ... + z - 1 = \frac{z(z-1)}{2}$. As we want them to be equal, $z = \frac{z(z-1)}{2} \Rightarrow z = 3$. So every $n = p^3$ (where p is prime) is f-perfect for the function $\tau(n) - 1$. Now we will look at the case when n is not a power of a prime. We know that the following equation holds : $\tau(n) - 1 = \sum_{d|n,d<n}(\tau(d) - 1)$. We add $\tau(n) - 1$ to both sides and therefore $2(\tau(n) - 1) = \sum_{d|n}(\tau(d) - 1)$. Furthermore, by adding $\tau(n)$ we reach $3\tau(n) - 2 = \sum_{d|n}(\tau(d))$. Now let us assume that there is more than one prime divisor, which has a power greater than 1. Let those be p and q and let $n = p^{t_1}q^{t_2}x$ $(t_1, t_2 > 1)$. Then we have that $\sum_{d|n}(\tau(d)) \geq \tau(n) + (\tau(\frac{n}{p}) + \tau(\frac{n}{p^{t_1}})) + (\tau(\frac{n}{q}) + \tau(\frac{n}{q^{t_2}})) = \tau(n) + \tau(n) + \tau(n) = 3\tau(n)$. This means that $3\tau(n) - 2 \geq 3\tau(n)$, which is not possible. So there is at most only 1 divisor with a power over 1 (let this divisor be p). Let $n = p^t x$, therefore $\tau(n) = (t + 1)\tau(x)$.

Lemma 1. *We have that* $\sum_{d|n}(\tau(d)) = \frac{(t+1)(t+2)}{2} \sum_{d|x}(\tau(d))$.

Proof. Each divisor of n can be derived from the divisors of x, multiplied by 1, p, p^2,…,p^t and τ is a multiplicative function so: $\sum_{d|n}(\tau(d)) = \tau(1)\sum_{d|x}(\tau(d)) + \tau(p)\sum_{d|x}(\tau(d)) + \cdots + \tau(p^t)\sum_{d|x}(\tau(d)) = \tau(1)\sum_{d|x}(\tau(d)) + 2\sum_{d|x}(\tau(d)) + \cdots + (t + 1)\sum_{d|x}(\tau(d)) = \frac{(t+1)(t+2)}{2}\sum_{d|x}(\tau(d))$ □

Using this lemma we can prove that if an integer A has a canonical representation $A = p_1p_2 \dots p_z$, $\sum_{d|A}(\tau(d)) = \frac{2^z3^z}{2^z} = 3^z$.

From the lemma $\sum_{d|n}(\tau(d)) = \frac{(t+2)(t+1)}{2}\sum_{d|x}(\tau(d))$, therefore $3(t + 1)\tau(x) - 2 = \frac{(t+2)(t+1)}{2}\sum_{d|x}(\tau(d))$. We know that $\tau(x) = 2^{z-1} \Rightarrow 3(t + 1)2^{z-1} - 2 = \frac{(t+2)(t+1)}{2}\sum_{d|x}(\tau(x)) \Rightarrow 3(t + 1)2^{z-1} - 2 = \frac{(t+2)(t+1)3^{z-1}}{2} \Rightarrow (t + 1)((t + 2)3^{z-1} - 3 \times 2^z) = -4$. As $t + 2 \geq 3$ we can speculate that for $z > 2 \Rightarrow 3^{z-1} > 2^z \Rightarrow (t + 2)3^{z-1} - 3 \times 2^z > 0$ and $(t + 1) > 0 \Rightarrow$ the equation has no solution. For $z = 2$ $(t + 1)(3(t + 2) - 3 \times 4) = -4$ or t is not an integer. We have solved the problem for $f(n) = \tau(n) - 1$. Now let us observe some cases for $f(n) = \tau(n) + k$.

If n is a power of a prime number ($n = p^z$), $\tau(n) = z + 1$. $\sum_{d|n,d<n} \tau(d) = 1 + 2 + \cdots + z = \frac{z(z+1)}{2}$. Therefore, for the function $f(n) = \tau(n) + k$, we want the following equation to be true: $z + 1 + k = \frac{z(z+1)}{2} + (\tau(n) - 1)k$. Solving this equation, we get to $k = -\frac{(z-2)(z+1)}{2(z-1)}$. As we want k to be an integer, we know that $z - 1|z + 1$ (bearing in mind that $z - 1$ and $z - 2$ are co-prime numbers). Therefore z can be only 2 or 3. If $z = 2$, $k = 0$ and this leads us back to **Problem 1.a)** and if $z = 3$, $k = -1$ and we solved this case above.

1) If $k > 0$ there are no f-perfect numbers for the function $f(n) = \tau(n) + k$.

Proof. We add $\tau(n)$ to both sides of the equation and we get to the following thing

$$\sum_{d|n} \tau(d) = (2-k)\tau(n) + 2k.$$

As n is composite number, the left side is bigger than 5. This leads to $k \leq 0$. □

2) If $k = 0$ we proved that $n = p^2$ (p is prime) are f-perfect for the function $f(n) = \tau(n)$. **Problem 1.a)**

3) If $k = -1$ we proved that only the numbers that are $n = p^3$ (p is prime) are f-perfect for the function $f(n) = \tau(n) - 1$. **Problem 1.b)**

4) If $k = -3t$ we found out that only the numbers $n = p^{2t-1}q^2r$ are f-perfect for the function $f(n) = \tau(n) - 3t$ (we wrote a program that confirmed this result although we did not manage to prove it).

5) For every other value for $k \in \mathbb{Z}$ we found out that there are no f-perfect numbers.

Problem 2. *Find all* f-perfect *numbers* n*, where* $f(n) = \varphi(n)$ *is Euler's totient function.*

Solution: We can assume that $n \geq 2$ as $n = 1$ is not f-perfect for the Euler's totient function. From the statement of the problem we know that $\varphi(n) = \sum_{d|n, d \leq n-1} \varphi(d)$. We add $\varphi(n)$ to both sides and using the following theorem $\sum_{d|n} \varphi(n) = n$ we come to the conclusion that $n = 2\varphi(n)$. If 2 divides the right side, we know that 2 divides the left one as well. We know that: $\varphi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \dots (1 - \frac{1}{p_k})$ (where p_i are the prime divisors of n). Therefore as $2|n$ we know that $\varphi(n) \leq \frac{n}{2}$ (both sides are equal only when the only prime divisor of n is 2). Since we have $2\varphi(n) = n$ we know that 2 is the only prime divisor of n. Therefore $n = 2^k$, $k \in \mathbb{N}$. We can easily check that if $n = 2^k$, n is f-perfect for the Euler's totient function, bearing in mind that $2^{k+1} = 2^k + 2^{k-1} + \dots + 2^1 + 2^0 + 1$.

Problem 3. *a) Prove that if* k *is a natural number, such that* $2^{k+1} - 2k - 1$ *is prime, then* $n = 2^k(2^{k+1} - 2k - 1)$ *is* f-perfect *for* $f(n) = n - 1$.

b) Find similar sufficient conditions for f-perfectness *for other polynomial functions of degree* 1 *such as* $f(n) = n - 2$ *is* f-perfect *for* $f(n) = n + 1$

Solution: We will use the following famous theorem:

Theorem 2. *The sum of the divisors of* n *(*$\sigma(n)$*) is equal to*

$$\frac{(p_1^{l_1+1} - 1)(p_2^{l_2+1} - 1) \dots (p_k^{l_k+1} - 1)}{(p_1 - 1)(p_2 - 1) \dots (p_k - 1)},$$

where the canonical representation of n *is* $p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$.

a) On the one hand $f(n) = 2^k(2^{k+1} - 2k - 1) - 1 = 2^{2k+1} - 2^{k+1}k + 2^k - 1$. On the other hand using **Theorem 2** we have that

$$\begin{aligned} \sum_{k|n, k \leq n-1} k &= \frac{(2^{k+1} - 1)((2^{k+1} - 2k + 1)^2 - 1)}{(2-1)((2^{k+1} - 2k + 1) - 1)} - n \\ &= \frac{2^{k+1} - 1}{2^{k+1} - 2k} - n \\ &= 2^{2k+1} - 2^{k+1}k + 2^k + 2k. \end{aligned}$$

As we have the function $f(n) = n - 1$, we have to subtract 1 for every divisor of n, which is smaller than n . Therefore, $\sum_{k|n, k\leq n-1} f(k) = \sum_{k|n,k\leq n-1} k - \tau(n) - 1 = 2^{2k+1} - 2^{k+1}k + 2^k + 2k - 2k - 1 = 2^{2k+1} - 2^{k+1}k + 2^k - 1$.

b) We present the following

Conjecture 1. *If* $n = 2^k(2^{k+1} + 2kl - 1)$ *is prime then* n *is* f*-perfect for the function* $f(n) = n - l$.

On the one hand $f(n) = 2^{2k+1} + 2^{k+1}kl - 2^k - l$. On the other hand $\sum_{k|n,k\leq n-1} f(k) = \sum_{k|n,k\leq n-1} k - (\tau(n) - 1) \times l = (2^{k+1} - 1)(2^{k+1} + 2kl) - 2^{2k+1} + 2^{k+1}kl - 2^k - l - 2(k + 1)l + l = 2^{2k+1} + 2^{k+1}kl - 2^k - l$.

The proof is identical to the one in **a)**. Using **Theorem 2**.

Problem 4. *Let* $f(n) = ln(n)$. *Find all* f-perfect *numbers* n.

Solution: We know that $ln(n) = \sum_{d|n} ln(d)$. Let all divisors of n be d_1, d_2, d_3,...,d_k. Therefore $t(n) = k + 1$ and $\sum_{d|n} ln(n) = ln(1) + ln(d_1) + ln(d_2) + \cdots + ln(d_k)$. We will you use that $ln(a) + ln(b) = ln(ab)$, so $\sum_{d|n} ln(n) = ln(1 \times d_1 \times d_2 \times d_3 \times \cdots \times d_k)$. We know that $\pi(n) = n^{\frac{\tau(n)}{2}}$ (where $\pi(n)$ is the multiplication of all divisors of n (including n)). Therefore $\sum_{d|k} ln(n) = \frac{\pi(n)^{\frac{\tau(n)}{2}}}{n}$, so $\frac{\tau(n)}{2} = n^2$. From the last equation we find that $\tau n = 4$. This is possible only if $n = p^3$ (p is prime number) or $n = pq$ (p and q are prime numbers). We easily check that if $n = p^3$ or $n = pq$, n is f-perfect for the function $f(n) = ln(n)$.

Problem 5. *Let* $f(n) = (-1)^n$. *Find all* f-perfect *numbers* n. *Study the general case that* $f(n) = w^n$, *where* $w \in \mathbb{C}$ *is a root of unity.*

Solution:

5.1. First we will study the case where $f(n) = (-1)^n$.

5.1.1. Let n be an odd number. $f(n) = -1$, since $2 \nmid n$. Analogically $\sum_{d|n,d\leq n-1} = -1(\tau(n) - 1)$ (the number of the divisors of n (without n) is $\tau - 1$ and they are all odd). As $\sum_{d|n,d\leq n-1} = -1(\tau(n) - 1)$, we know that $\tau(n) = 2$. This is possible only when n is a prime number.

5.1.2. Let n be $2^k t$ and t is an odd number. $f(n) = (-1)^n = (-1)^{2^k t} = 1$.

In the sum, we have exactly $k\tau(t) - 1$ even numbers and $\tau(t)$ odd numbers. Therefore $\sum_{d|n,d\leq n-1} = k\tau(t) - 1 - \tau(t) = (k - 1)\tau(t) - 1 = 1$ (as it is equal to $f(n)$). This is equal to $(k - 1)\tau(t) = 2$.We will mention $\tau(n) \geq 1$. The equation has 2 solutions:

1) $\tau(n) = 1$ and $k = 3$. This means that $t = 1$ and $n = 8$.

2) $\tau(n) = 2$ and $k = 2$. This means that t is prime and $n = 4t$.

To sum up, we proved that n is f-perfect for the function $(-1)^n$ if $n = 8$, p, $4p$ (p is prime and $p > 2$).

Problem 6. *Let* $f(n) = \binom{2012}{n}$. *Find all* f-perfect *number* n. *Study the general case where* 2012 *is replaced by a natural number* m.

Solution: Our aim will be to prove that n is f-perfect for the function $\binom{m}{n}$, if and only if n is prime and $m = n + 1$.

If $n > m$ we can easily see that n is not f-perfect, as $f(n)$ will be smaller than $\sum_{d|m,1\leq d\leq n-1} f(d)$.

First of all, we will solve the case where $m = 2012$. If $n = 1$, 2011 or 2012 we can easily check whether n is f-perfect or not. We see that $n = 2011$ is f-perfect (2011 is prime , and $m = 2012 = 2011 + 1 = n + 1$). Now let's assume that $2 \leq n \leq 2010$. Let all divisors ≥ 2 of n be p_i (including n). Since they divide n , they are between 2 and 2010. Therefore, using the formula $\binom{2012}{p_i} = \frac{2012!}{p_i!(2012-p_i)!}$ we see that $2011|\binom{2012!}{p_i}$ (as 2011 is prime and $2011|2012!$ and $2011 \nmid p_i!(2012-p_i)!$). We have $f(n) = \binom{2012}{n} = \sum_{d|m, 1 \leq d \leq n-1} f(d) = \binom{2012}{1} + \binom{2012}{p_1} + \binom{2012}{p_2} + \cdots + \binom{2012}{p_i}$. Now as the left side is divisible by 2011, the right one should be as well. However as we mentioned above $2011|\binom{2012!}{p_i}$ and therefore 2011 should delete $\binom{2012}{1}$, but we can easily see that this is not true. Therefore if n is between 2 and 2010, n is not f-perfect for the function $f(n) = \binom{2012}{n}$.

Now if n is prime , 1 is the only divisor of n. This means that $f(n) = f(1)$. Furthermore, $\binom{m}{n} = \binom{m}{1}$. Therefore m is exactly $n + 1$.

Problem 7. *Consider other arithmetic functions f and find sufficient and/or necessary conditions for a number to be f-*perfect*.*

Solution:

7.1. Directly from the fourth problem we see that for every function $f = \log_k(n)$, where $k > 0$, $k \neq 1$, the only numbers that are f-perfect are the numbers which are exact third power of a prime or which have only 2 divisors of power 1. In other words $n = pq$ or $n = p^3$ where p and q are prime numbers.

7.2. Let $f(n) = \mu(n)$ where $\mu(n)$ is $M\ddot{o}bius's$ function. It is obvious that $n = 1$ is not f-perfect and we know that when $n > 1$, $\sum_{d|n} \mu(n) = 0$. Therefore $2\mu(n) = 0$ $\Rightarrow n = k^2 m$ are all f-perfect numbers ($k > 1$, $m \geq 1$).

7.3. Let $f(n) = \delta(n)$ be Mangoldt's function.

$$\delta(n) = \begin{cases} \ln p, & \text{if } n > 1 \text{ and } n \text{ is power of a prime,} \\ 0, & \text{if } n \text{ is not power of a prime.} \end{cases}$$

We must mention that this function is not multiplicative. However it is a well-known fact that $\sum_{d|n} \delta(d) = \log n$.

Proof. If $n = 1$, $\sum_{d|n} \delta(d) = \delta(1) = 0 = \log 1$. Now let $n > 1$ and $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. In the sum the only addends that are different from zero are $\delta(p_i^{l_i}) = \log p_i$. Therefore $\sum_{d|n} \delta(n) = (\delta(p_1) + \delta(p_1^2) + \cdots + \delta(p_1^{k_1})) + \cdots + (\delta(p_r) + \delta(p_r^2) + \cdots + \delta(p_r^{k_r})) = k_1 \log p_1 + k_2 \log p_2 + \cdots + k_r \log p_r = \log p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} = \log n$. □

Now in our problem, we just have to add $\delta(n)$ to both sides of the equation and we come to the following equation : $2\delta(n) = \ln n$. This means that $n = p^k$ and as $\ln p^2 = \ln n$, $k = 2$. Therefore the f-perfect numbers for the Mangoldt's function are $n = p^2$.

Problem 8. *Recall that a pair (m,n) of positive integers is said to be amicable if $n = \sum_{d|m} d$ and $m = \sum_{d|n} d$.*

An example is (220,284). For an arithmetic function f, give a reasonable definition for a pair (m,n) of positive integers to be f*-amicable*. For various arithmetic functions f, find f*-amicable* pair or prove that no f*-amicable* pairs exist.

Solution: In order for a pair (m, n) to be f-amicable the following must be true:

$$f(n) = \sum_{d|m, d<m} (f(d)) \tag{0.3}$$

and

$$f(m) = \sum_{d|n, d<n} (f(d)) \tag{0.4}$$

If we add $f(n)$ to both sides of (0.4), $f(m) = \sum_{d|n,d<n}(f(d))$ we reach the equation $f(n)+f(m) = \sum_{d|n,d<n}(f(d))+f(n)$. If we replace $f(n)$ with its equal on the right side we reach $\sum_{d|n,d<m}(f(d)) + f(m) = \sum_{d|n,d<n}(f(d)) + f(n)$ or $\sum_{d|n}(f(d)) = \sum_{d|m}(f(d))$.

8.1. If $f(n) = A\tau(n) + B$ where $A, B \in \mathbb{N}$ then the solutions of this equation is every pair (m, n) in which if the canonical representation of m is $m = p_1^{a_1} p_2^{a_2} \dots p_z^{a_z}$, n must be

$$q_1^{a_1} q_2^{a_2} \dots q_z^{a_z} \tag{0.5}$$

As we can see there is no significance in the actual value of the primes (as the function τ only takes into account their exponents). So in order for us to solve the equation we just need to solve (0.3) and every pair with the necessary properties given in (0.5) will be a solution.

References

[1] Bulgarian journal "Matematika", issue 6, 2009

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