4^{th} International Tournament of Young Mathematicians

Belarus

Problem 9. Random games

Abstract

This problem is about random graphs and games on them. In this paper we considered two-player games on Galton-Watson trees with offspring distribution μ (GW_μ tree), where each player move a token from a vertex to a neighbouring one, never returning to already visited vertices, and a player, who is unable to move, loses (game $\mathcal{G}_1$) or wins (game $\mathcal{G}_2$).

In the first part of the paper we studied the probability of GW_μ tree to be finite (p_{ext}). We proved that this probability is the least solution to $F(x) = x$, where $F(x)$ is the generating function of μ. We also found conditions to have $p_{\text{ext}} = 1$, that solved question 1 of the initial statement of the problem.

Then we considered the probabilities of win, lose, draw of the first player in games $\mathcal{G}_1, \mathcal{G}_2$ ($w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$). We studied bounds and inequalities for these values (questions 2 and 3).

We considered also particular cases of distributions, for which more precise results can be obtained. We studied the Poisson distribution, and proved that functions $w(\lambda), l(\lambda), d(\lambda), \tilde{w}(\lambda), \tilde{l}(\lambda), \tilde{d}(\lambda)$ are infinity-differentiable with respect to λ in case $\lambda \neq e$ and continuous in each point $\lambda > 0$, but not differentiable in point $\lambda = e$ (question 4). Also we studied the case of the geometric distribution, found explicit formulas for probabilities of win, lose and draw.

In the last part of the paper we considered the strategy for $\mathcal{G}_1, \mathcal{G}_2$ and studied these games on particular graphs.

Contents

Initial statement and definitions

Given a graph G and a vertex v, consider the following two-player games. Both games begin with a token placed on v, and the players alternately move the token from its current vertex to a neighbouring one, never returning to already visited vertices. In the game $\mathcal{G}_1$, a player unable to move the token in this way loses. In the game $\mathcal{G}_2$, a player who is unable to move wins. The games may be a win for the first player, a win for the second one or else a draw (if a graph is infinite). The goal of this problem is to play such games on random graphs.

Let $\mu = (\mu_i)_{i \geq 0}$ be a sequence of non-negative real numbers such that

$$\sum_{i=0}^{\infty} \mu_i = 1 \text{ and } \mu_0 + \mu_1 < 1$$

The Galton-Watson tree with offspring distribution μ, or shortly the GW_μ tree, is a random plane tree defined as follows. Start with a single vertex (the ancestor) having a random number of children distributed according to μ, which means that the probability of having i children is μ_i. Then each of its children has itself an independent random number of children distributed according to μ, and so on. The genealogical tree of this population is the GW_μ tree. It is a random finite or infinite tree. We denote by p_{ext} the probability that the GW_μ tree is finite.

1. For $0 \leq x \leq 1$, consider the series

$$F(x) = \sum_{i=0}^{\infty} \mu_i x^i$$

(a) Show that p_{ext} is the least solution to the equation $F(x) = x$ in the interval $[0, 1]$.

(b) Give a necessary and sufficient condition in order to have $p_{ext} = 1$.

Two players play the games $\mathcal{G}_1$, $\mathcal{G}_2$ on the GW_μ tree starting from the ancestor. Set:

$$\begin{aligned} w &= \mathbb{P}\big(\text{first player wins the game } \mathcal{G}_1\big), & \tilde{w} &= \mathbb{P}\big(\text{first player wins the game } \mathcal{G}_2\big) \\ l &= \mathbb{P}\big(\text{first player loses the game } \mathcal{G}_1\big), & \tilde{l} &= \mathbb{P}\big(\text{first player loses the game } \mathcal{G}_2\big) \\ d &= \mathbb{P}\big(\text{draw in the game } \mathcal{G}_1\big), & \tilde{d} &= \mathbb{P}\big(\text{draw in wins the game } \mathcal{G}_2\big) \end{aligned}$$

2. Estimate $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$.

3. Find inequalities between $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$.

4. For $\lambda > 0$ and $i \geq 0$, set:

$$\mu_i^{(\lambda)} = \frac{\lambda^i}{i!} e^{-\lambda}.$$

Let $w^{(\lambda)}, l^{(\lambda)}, d^{(\lambda)}, \tilde{w}^{(\lambda)}, \tilde{l}^{(\lambda)}, \tilde{d}^{(\lambda)}$ be defined as previously. Are these functions continuous with respect to λ? Differentiable with respect to λ?

5. Investigate games $\mathcal{G}_1$ and $\mathcal{G}_2$ on other types of random graphs.

1 Probability of finite tree

1.1 Properties of F

Firstly consider the case when $\mu_0 = 0$. Since each vertex has a non-zero number of children, it's clear that $p_{\text{ext}} = 0$ and $d = \tilde{d} = 1$. So next we assume that $\mu_0 > 0$.

As $\mu_1 + \mu_0 < 1$ we have $F'(x) > 0$ and $F''(x) > 0$ for all $x \in]0,1]$, Hence $F(x)$ and $F'(x)$ are strictly increasing on $[0,1]$ functions. Since $\forall\, x \in]0,1]$ there holds $F''(x) > 0$, so $F(x)$ is convex function. Consider two cases.

Lemma 1.1. *If $F'(1) \leq 1$ then F has exactly one fixed point on $[0,1]$, which is 1.*

$\square$ Suppose there exist at least two fixed points in this case. Then function $H(x) = x - F(x)$ has at least two zeroes ξ_1, ξ_2. Then by Rolle's theorem $H'(x) = 1 - F'(x)$ has a zero ξ_0, with $\xi_1 < \xi_0 < \xi_2$ and $F'(\xi_0) = 1$. Since $F'(1)$ is increasing and $F'(1) \leq 1$ we have $\xi_0 = 1$. But then $\xi_2 > \xi_0 > 1$, this is impossible. As $\sum_{i=0}^{\infty} \mu_i = 1$, we have $F(1) = 1$ and 1 is the fixed point. $\blacksquare$

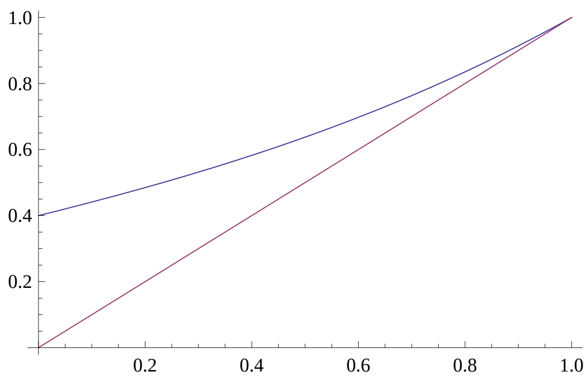

Figure 1: $F'(1) \leq 1$ one fixed point

Lemma 1.2. *If $F'(1) > 1$, then F has two fixed points on $[0,1]$.*

$\square$ Suppose there exist three or more solutions to $F(x) = x$. Then function $H(x) = x - F(x)$ has at least three zeroes. Then by Rolle's theorem $H'(x) = 1 - F'(x) = 0$ has at least two solutions. But $F'(x)$ is strictly increasing and may take each value only once. Thus $H(x)$ has at most one zero.

Clearly 1 is a fixed point. Since $H(0) \leq 0$, $H(1) = 0$, $H'(0) > 0$, $H'(1) < 0$, so there exists $\alpha \in [0,1[$, such that $H(\alpha) = 0$. Thus F has two fixed points. $\blacksquare$

1.2 p_{ext} description

Theorem 2. *p_{ext} is the least solution to $F(x) = x$ on $[0,1]$.*

$\square$ Define sequence $\left(p_k\right)_{k\geq 0}$ as the the following. Let p_k be the probability that GW_μ has not greater than k generations not including the main ancestor. Then it's clear that $p_0 = \mu_0$. Using the law of total probability we have

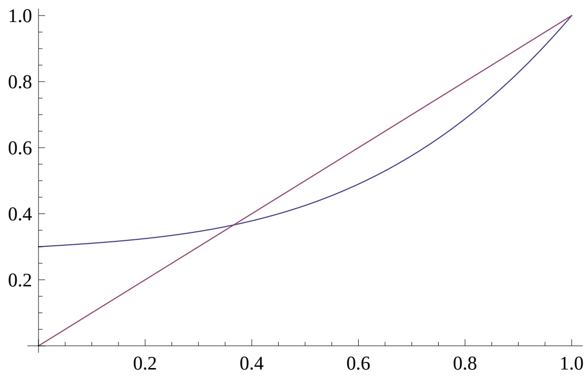

Figure 2: $F'(1) > 1$ two fixed points

$$p_k = \sum_{i=0}^{\infty} \mu_i (p_{k-1})^i \iff p_k = \underbrace{F(...F(F}_{\text{k times}}(\mu_0)...) = F^k(\mu_0). \tag{1}$$

Since p_{ext} is the probability that GW_μ is a finite tree, we have $p_{\text{ext}} = \lim_{k\to\infty} p_k$. As p_k is increasing and bounded, so this limit exists. Then p_{ext} is the result of infinite number of iterations of F starting with μ_0. Denote this iterative process by Ψ_F.

Consider the least fixed point α of $F(x)$ on $[0,1]$. Since F is increasing, so $\mu_0 = F(0) < F(\alpha) = \alpha$. Also $\forall\, x \in [0,\alpha]$ there holds $F(x) \leq F(\alpha) = \alpha$. And hence p_k can't leave the interval $[0,\alpha]$. As $F(p_{k-1}) = p_k$ and F is continuous on $[0,1]$, we have $p_{\text{ext}} = \lim_{k\to\infty} p_k = \alpha$. ■

1.3 Conditions to have $p_{\textbf{ext}} = 1$

Corollary 2.1. *$p_{ext} = 1$ iff $F'(1) \leq 1$.*

□ From lemma 1.1, in case $F'(1) \leq 1$, F has only one fixed point $\alpha = 1$, which is clearly p_{ext}.

From lemma 1.2, in case $F'(1) > 1, F$ has two fixed points α_1, α_2, with $\alpha_1 < \alpha_2 = 1$ and from theorem 2 Ψ_F converge to α_1 which is p_{ext} and < 1.

Thus to have $p_{\text{ext}} = 1$ it's necessary and sufficient to have $F'(1) \leq 1$.■

2 Estimations and inequalities for $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$

2.1 Relations between $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$

Denote by $w_k, l_k, \tilde{w}_k, \tilde{l}_k$ respectively the probability that the first player wins/loses in game $\mathcal{G}_1/\mathcal{G}_2$ on tree GW_μ not later than k^{th} move. Using the total probability law, we have:

$$w_k = \sum_{i=0}^{\infty} \mu_i \big(1 - (1 - l_{k-1})^i\big) \ , \ l_k = \sum_{i=0}^{\infty} \mu_i w_{k-1}^i \text{ and} \tag{2}$$
$$\tilde{w}_k = \sum_{i=0}^{\infty} \mu_i \big(1 - (1 - \tilde{l}_{k-1})^i\big) \ , \ \tilde{l}_k = \sum_{i=0}^{\infty} \mu_i \tilde{w}_{k-1}^i$$

or in other words $w_k = 1 - F(1 - l_{k-1}), l_k = F(w_k)$, for $k > 0$ and $w_0 = 0, l_0 = \mu_0$ and $\tilde{w}_k = 1 - F(1 - \tilde{l}_{k-1}), \tilde{l}_k = F(\tilde{w}_k)$ with $\tilde{w}_0 = \mu_0, \tilde{l}_0 = 0$. Denote by ld the probability that the first player won't win and by ld_k the probability that it will happen not later than the k^{th} move, $ld_k = 1 - w_k$. We have $\lim_{k\to\infty} w_k = w$, $\lim_{k\to\infty} l_k = l$ and $\lim_{k\to\infty} ld_k = ld$, since all these sequences are bounded and increasing, so the limits exist.

Since $F(x)$ is continuous on $[0, 1]$, we have the system:

$$\begin{cases} w &= 1 - F(1 - l), \\ l &= F(w). \end{cases} \tag{3}$$

Let $x = l, y = dl = 1 - w$. Then we can rewrite system (3)

$$\begin{cases} y &= F(1 - x), \\ x &= F(1 - y). \end{cases} \tag{4}$$

2.2 Estimates for $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$

To get estimations for $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$ consider two cases.

1. $F'(1) \le 1$. We have $d = 0$, let $z = 1 - x = w$, then $1 - z = F(z)$. We obtain

$$\mu_0 + \mu_1 z \le 1 - z = \sum_{i=0}^{\infty} \mu_i z^i \le \mu_0 + \mu_1 z + (1 - \mu_0 - \mu_1) z^2.$$

We have $z \le \dfrac{1-\mu_0}{1+\mu_1}$ and $(1 - \mu_0 - \mu_1)z^2 + (1 + \mu_1)z + \mu_0 - 1 \ge 0$ and hence

$$\frac{-1 - \mu_1 + \sqrt{(1+\mu_1)^2 + 4(1-\mu_0)(1-\mu_0-\mu_1)}}{2(1-\mu_0-\mu_1)} \le \ w = z \ \le \frac{1-\mu_0}{1+\mu_1}, \tag{5}$$
$$\frac{\mu_0+\mu_1}{1+\mu_1} \le \ l = x \ \le \frac{3 - 2\mu_0 - \mu_1 - \sqrt{(1+\mu_1)^2 + 4(1-\mu_0)(1-\mu_0-\mu_1)}}{2(1-\mu_0-\mu_1)}.$$

2. $F'(1) > 1$. Denote by $z = 1 - x = 1 - l$ and $q = 1 - y = w$. Note that $z \ge q$. We have

$$\begin{cases} 1 - q &= F(z) &= \sum_{i=0}^{\infty} \mu_i z^i, \\ 1 - z &= F(q) &= \sum_{i=0}^{\infty} \mu_i q^i, \end{cases} \iff \begin{cases} \mu_0 + \mu_1 z &\le& 1 - q &\le& \mu_0 + (1-\mu_0) z, \\ \mu_0 + \mu_1 q &\le& 1 - z &\le& \mu_0 + (1-\mu_0) q. \end{cases}$$

There holds

$$\begin{cases} (1-\mu_0)(1-z) &\le& q &\le& 1 - \mu_0 - \mu_1 z, \\ (1-\mu_0)(1-q) &\le& z &\le& 1 - \mu_0 - \mu_1 q. \end{cases} \tag{6}$$

Firstly estimate z. Let z_1 be the solution to $z = (1-\mu_0)(1-z)$, and z_2 be the solution to the system

$$\begin{cases} q &= (1-\mu_0)(1-z), \\ z &= 1-\mu_0-\mu_1 q. \end{cases}$$

Then we have $z_1 \le z \le z_2$. We get

$$1-\mu_0 = \frac{z_1}{1-z_1} \iff \frac{1}{1-\mu_0} = \frac{1}{z_1} - 1 \iff z_1 = \frac{1-\mu_0}{2-\mu_0},$$

$$z_2 = (1-\mu_0)(1-\mu_1+\mu_1 z_2) \iff z_2 = \frac{1-\mu_0-\mu_1+\mu_0\mu_1}{1-\mu_1+\mu_0\mu_1}.$$

Thus the following bounds hold

$$\frac{\mu_0}{1-\mu_1+\mu_0\mu_1} \le x = l \le \frac{1}{2-\mu_0}. \tag{7}$$

Find bounds for q. We have $q_1 \le q \le q_2$, where q_2 is the solution to $1-\mu_0-\mu_1 q = q$ and q_1 is the solution to the system

$$\begin{cases} q &= (1-\mu_0)(1-z), \\ z &= 1-\mu_0-\mu_1 q. \end{cases}$$

We obtain

$$q_1 = (1-\mu_0)(\mu_0+\mu_1 q_1) \iff q_1 = \frac{\mu_0(1-\mu_0)}{1-\mu_1+\mu_0\mu_1}, \qquad q_2 = \frac{1-\mu_0}{1+\mu_1}.$$

Consequently,

$$\frac{\mu_0(1-\mu_0)}{1-\mu_1+\mu_0\mu_1} \le \ q = w \ \le \frac{1-\mu_0}{1+\mu_1}, \quad 0 \le \ d = z - q \ \le \frac{1-\mu_1+\mu_0\mu_1-2\mu_0+\mu_0^2}{1-\mu_1+\mu_0\mu_1}. \tag{8}$$

2.3 Inequalities between $w, l, d, \tilde{w}, \tilde{l}, \tilde{d}$

From system (3) we have $l = F(w) = \sum_{i=0}^{\infty} \mu_i w^i$, and hence we can obtain

$$\mu_0 + \mu_1 w + \mu_2 w^2 \le \ l \ \le \mu_0 + \mu_1 w + (1-\mu_0-\mu_1)w^2.$$

Also we have $d = (\sum_{i=0}^{\infty} \mu_i (1-l)^i) - l$, so

$$\mu_0 + \mu_1 - (\mu_1+1)l \le \ d \ \le \mu_0 + (1-\mu_0)(1-l) - l = 1 - (2-\mu_0)l.$$

Remark. Note that we got only estimates and inequalities for w, l, d, but since to get them we use only system (4), which holds also for $\tilde{w}, \tilde{l}, \tilde{d}$, these results are true for $\tilde{w}, \tilde{l}, \tilde{d}$.

3 Poisson distribution

Consider the distribution defined as $\mu_i = \mu_i(\lambda) = \frac{\lambda^i}{i!}e^{-\lambda}$ with $\lambda > 0$.

3.1 Properties of w, l, d

Let $x = l, y = dl = 1 - w$. Then we can rewrite system (4), using $F(x) = e^{-\lambda(1-x)}$ in the following way:

$$\begin{cases} y &= F(1-x), \\ x &= F(1-y), \end{cases} \iff \begin{cases} y &= e^{-\lambda x}, \\ x &= e^{-\lambda y}, \end{cases} \iff \begin{cases} y &= e^{-\lambda x}, \\ y &= \frac{-\ln x}{\lambda}. \end{cases} \tag{9}$$

From system (9) we have x as a solution to

$$x = \exp\left(-\lambda e^{-\lambda x}\right). \tag{10}$$

Lemma 3.1. *If $\lambda \in (0, e]$, then equation $x = \exp\left(-\lambda e^{-\lambda x}\right)$ has unique solution x_0, which is the solution to $\ln x_0 + \lambda x_0 = 0$.*

□ Equation $x = \exp\left(-\lambda e^{-\lambda x}\right)$ is equivalent to system (9). Equation $\ln x + \lambda x = 0$ has only one solution x_0, and pair (x_0, x_0) is a solution to the system.

System (9) is symmetric (i.e. if (x_0, y_0) is a solution then (y_0, x_0) is a solution too). It's easy to see that system (9) has only one solution (x_0, y_0) with $x_0 = y_0$.

Let $G(x) = x - \exp\left(-\lambda e^{-\lambda x}\right)$.

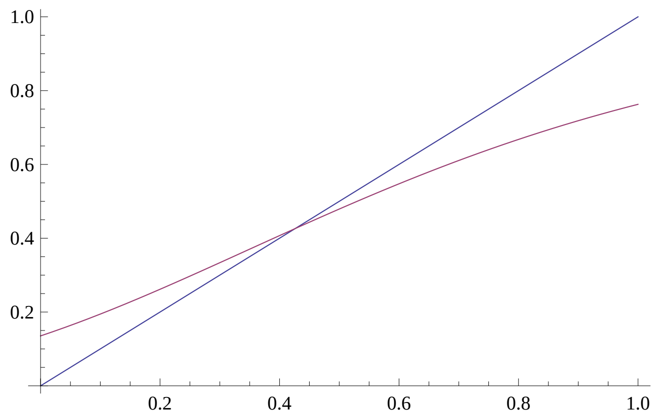

Figure 3: $\lambda \le e$, one intersection of x with $\exp\left(-\lambda e^{-\lambda x}\right)$

Suppose that G has more than one zero, i.e. G has at least three zeroes. Hence by Rolle's theorem function $G'(x) = 1 - \lambda^2 \exp\left(-\lambda x - \lambda e^{-\lambda x}\right)$ has at least two zeroes, and function $G''(x) = -\lambda^2\big(-\lambda + \lambda^2 e^{-\lambda x}\big)\exp\left(-\lambda x - \lambda e^{-\lambda x}\right)$ has at least one zero, which is between zeroes of $G'(x)$.

As $\lambda^2 \exp\left(-\lambda x - \lambda e^{-\lambda x}\right) > 0$, we have $G''(x) = 0$ is equivalent to

$$-\lambda + \lambda^2 e^{-\lambda x} = 0 \iff e^{-\lambda x} = \lambda^{-1} \iff x = \frac{\ln \lambda}{\lambda}$$

and analogously $G''(x) > 0 \iff x > \frac{\ln \lambda}{\lambda}$.

Since $G'(\frac{\ln \lambda}{\lambda}) = 1 - \lambda^2 \exp(-\ln \lambda - 1) = 1 - \frac{\lambda}{e} \ge 0$ and $G''(x) > 0$ for $x > \frac{\ln \lambda}{\lambda}$, then function $G'(x)$ has no zeroes which are greater than $\frac{\ln \lambda}{\lambda}$. This is contradiction. ■

Lemma 3.2. *If $\lambda > e$, then equation $x = \exp\left(-\lambda e^{-\lambda x}\right)$ has three different roots $x_1 < x_2 < x_3$ on $[0, 1]$, which satisfy the following properties:*

1. $\ln x_2 + \lambda x_2 = 0$;

2. $x_2 < \frac{1}{e}$;

3. $\ln x_1 \ln x_3 < 1$.

□ If there are more than three solutions, then there are at least five solutions, then function $G''(x) = -\lambda^2\big(-\lambda+\lambda^2 e^{-\lambda x}\big)\exp\left(-\lambda x - \lambda e^{-\lambda x}\right)$ has at least three zeroes, that is impossible, since $G''(x)$ has only one zero, as was shown in the proof of the previous lemma.

Prove that there are exactly three solutions.

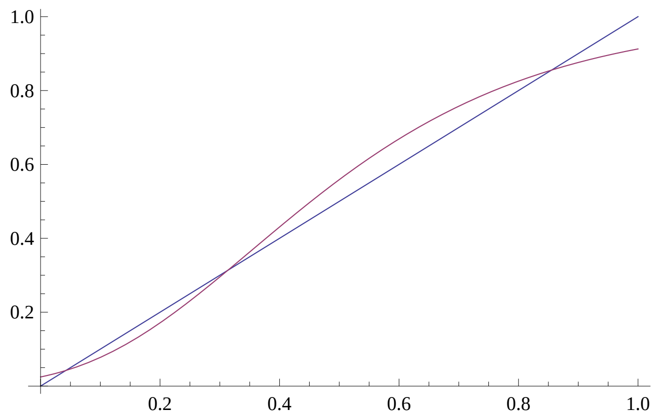

Figure 4: $\lambda > e$, three intersections of x with $\exp\left(-\lambda e^{-\lambda x}\right)$

Let x_2 be the unique solution to $\ln x + \lambda x = 0$.

Using $\ln x + \lambda x > \ln e^{-1} + xe \geq 0$ for any $x \geq \frac{1}{e}$ and $\lambda > e$, we have that $x_2 < \frac{1}{e}$.

Show that there exists at least one solution to $x = \exp\left(-\lambda e^{-\lambda x}\right)$ on interval $(0, x_2)$. Let $L(x) = \exp\left(-\lambda e^{-\lambda x}\right)$. Since $L(0) > 0$, it's enough to show that there exists such $x \in (0, x_2)$ that $x > L(x)$. It's the same that to show the existence of $x < x_2$, such that $-\lambda e^{-\lambda x} < \ln x$. From system (9) we have $\lambda = -\frac{\ln x_2}{x_2}$. Let

$$R(x) = \frac{\ln x_2}{x_2} e^{\frac{\ln x_2}{x_2} x} - \ln x.$$

Since $R(0) = 0$, it's sufficient to show that $R'(x_2) > 0$.

We obtain

$$R'(x_2) = \frac{\ln^2 x_2 - 1}{x_2} > 0 \iff \ln^2 x_2 - 1 > 0 \iff x_2 < \frac{1}{e}$$

that was proved.

From $R(x_2) = \frac{\ln x_2}{x_2} e^{\frac{\ln x_2}{x_2} x_2} - \ln x = 0$, $R(0) > 0$ and $R'(x_2) > 0$, we have that $L(x) = x$ has a solution on $(0, x_2)$.

Let's show that $\ln x_1 \ln x_3 < 1$.

Suppose a contrary, then $\ln x_1 \ln x_3 \geq 1$. Let $\alpha = \lambda x_1$ and $\beta = \lambda x_3$, then $\beta > \alpha$ and $\alpha\beta = x_1 x_3 \lambda^2 \geq 1$. From system (9) we have $\alpha - \ln\alpha = \beta - \ln\beta$. Function $h(x) = x - \ln(x)$ is decreasing on $(0,1)$ and increasing on $(1, +\infty)$. Then $\alpha < 1$ and $\beta \geq \frac{1}{\alpha}$, so we have

$$\alpha - \ln\alpha = \beta - \ln\beta \geq \frac{1}{\alpha} + \ln\alpha.$$

Hence $h_1(x) = \alpha - 2\ln\alpha - \frac{1}{\alpha} \geq 0$. Since $h_1'(x) = 1 - 2\left(\frac{1}{\alpha}\right) + \left(\frac{1}{\alpha}\right)^2 = \left(1 - \frac{1}{\alpha}\right)^2 > 0$ for $\alpha \in (0,1)$, so h_1 is increasing and $h_1(\alpha) < h_1(1) = 0$, this is a contradiction. ■

From system (2) we obtain $l_k = F(1 - F(1 - l_{k-2})) = G(l_{k-2})$ and $l_0 = \mu_0$, denote this iterative process by Ψ_G.

Theorem 4. *Iterative process Ψ_G converges to the least solution to equation $x = \exp\left(-\lambda e^{-\lambda x}\right)$.*

□ Let $\lambda \leq e$, then equation $x = G(x)$ has the unique root. Sequence $(l_{2n})_{n=0}^{\infty}$ is increasing and bounded and hence it has limit as $n \to \infty$, and obviously the limit is the unique fixed point of G.

Consider the case $\lambda > e$. Let α be the least fixed point of G. Let us prove that $\forall n \geq 0,\ l_{2n} \in [0, \alpha]$. It's sufficiently to check the following conditions.

1. $G([0,\alpha]) \subseteq [0,\alpha]$;

2. $\mu_0 \in [0,\alpha]$.

Condition 1. Since $G'(x) = \lambda^2 \exp\left(-\lambda x - \lambda e^{-\lambda x}\right) > 0$, function G is increasing. And hence for $x \leq \alpha$ there holds $0 \leq G(x) \leq G(\alpha) = \alpha$.

Condition 2. Suppose $\mu_0 > \alpha$. We know that $\mu_0 = F(0)$. Hence $F(0) > \alpha = F(\alpha) = F(x_1)$. But function F is increasing, and we obtain $0 > x_1$, that is contradiction.

Since $l_{2n} \in [0,\alpha]\ \forall n$, so $\lim\limits_{n\to\infty} l_{2n}$ also belongs to $[0,\alpha]$. ■

Remark. Since sequence $(l_k)_{k=0}^{\infty}$ is increasing and bounded, it has the limit, which is the same that the limit of $(l_{2n})_{n=0}^{\infty}$, and hence $\lim\limits_{k\to\infty} l_k = \alpha$, where α is the least solution to $x = \exp\left(-\lambda e^{-\lambda x}\right)$.

Corollary 4.1. Since $l(\lambda)$ is the least solution to $x = \exp\left(-\lambda e^{-\lambda x}\right)$, so $ld(\lambda)$ is the greater solution, and $d(\lambda) = ld(\lambda) - l(\lambda)$, $w(\lambda) = 1 - ld(\lambda)$.

3.2 Continuity and differentiability, $\lambda \neq e$

Let's show that functions $w(\lambda), l(\lambda), d(\lambda)$ are differentiable infinity many times in each point $\lambda \neq e$. To do it we'll use the foolowing theorem.

Implicit function theorem. [3, p. 246]

Let functions of three variables $F_1(x,y,\lambda), F_2(x,y,\lambda)$ are k times continuously differentiable in a neighbourhood of (x_0, y_0, λ_0), the Jacobian of the mapping

$$\varphi(x,y) \to (F_1(x,y,\lambda_0), F_2(x,y,\lambda_0))$$

is not zero in point (x_0, y_0), *i.e.*

$$\frac{\partial(F_1, F_2)}{\partial(x, y)}(x_0, y_0, \lambda_0) \neq 0, \text{ where } \frac{\partial(F_1, F_2)}{\partial(x, y)}(x_0, y_0, \lambda_0) = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix}$$

and $F_1(x_0, y_0, \lambda_0) = 0, F_2(x_0, y_0, \lambda_0) = 0$, *then there exist neighbourhoods* $V = V(\lambda_0)$ *and* $W = W(x_0, y_0)$, *such that there exists unique mapping* $\varphi\colon V \to W, \varphi = (\varphi_1, \varphi_2)$, *such that* $F_1(\varphi_1(\lambda), \varphi_2(\lambda), \lambda) \equiv 0$ *and* $F_2(\varphi_1(\lambda), \varphi_2(\lambda), \lambda) \equiv 0$ *for all* $\lambda \in V(\lambda_0)$ *and besides functions* $\varphi_1(x), \varphi_2(x)$ *are* k *times continuously differentiable in a neighbourhood of* $V(\lambda_0)$ *and derivatives* $\varphi_1'(\lambda_0), \varphi_2'(\lambda_0)$ *satisfy the system:*

$$\begin{cases} \varphi_1'(\lambda_0)\frac{\partial F_1}{\partial x}(\varphi(\lambda_0), \lambda_0) + & \varphi_2'(\lambda_0)\frac{\partial F_1}{\partial y}(\varphi(\lambda_0), \lambda_0) + & \frac{\partial F_1}{\partial \lambda}(\varphi(\lambda_0), \lambda_0) = 0, \\ \varphi_1'(\lambda_0)\frac{\partial F_2}{\partial x}(\varphi(\lambda_0), \lambda_0) + & \varphi_2'(\lambda_0)\frac{\partial F_2}{\partial y}(\varphi(\lambda_0), \lambda_0) + & \frac{\partial F_2}{\partial \lambda}(\varphi(\lambda_0), \lambda_0) = 0. \end{cases}$$

Theorem 5. *Functions* $w(\lambda), l(\lambda), d(\lambda)$ *are differentiable infinity many times for all* $\lambda \neq e$.

□ We have $F_1(x, y, \lambda) = \ln x + \lambda y, F_2(x, y, \lambda) = \ln y + \lambda x$. Show that for $\forall \lambda \neq e$ in point $(l(\lambda_0), ld(\lambda_0), \lambda_0)$ the conditions of implicit function theorem hold. Calculate the Jacobian:

$$\mathcal{J} = \frac{\partial(F_1, F_2)}{\partial(x, y)} = \begin{vmatrix} \frac{1}{x_0} & \lambda_0 \\ \lambda_0 & \frac{1}{y_0} \end{vmatrix} = \frac{1}{x_0 y_0} - \lambda_0^2. \tag{11}$$

Suppose $\mathcal{J} = 0$, i.e. $\lambda_0^2 x_0 y_0 = 1$. Then we have the system:

$$\begin{cases} \lambda_0^2 x_0 y_0 &= 1, \\ \ln x_0 &= -\lambda y_0, \\ \ln y_0 &= -\lambda x_0. \end{cases} \tag{12}$$

Set $\alpha = \lambda_0 x_0, \beta = \lambda_0 y_0$. From system (12) we have $\alpha - \beta = \ln \alpha - \ln \beta$, and then

$$\begin{cases} \alpha\beta &= 1, \\ \alpha - \beta &= \ln \alpha - \ln \beta. \end{cases} \tag{13}$$

From the first equation we get $\beta = \frac{1}{\alpha}$. Substituting to the second one can obtain: $\alpha - \frac{1}{\alpha} = \ln \alpha - \ln \frac{1}{\alpha} \iff h(\alpha) = \alpha - \frac{1}{\alpha} - 2\ln \alpha = 0, \alpha > 0$. Since $h'(\alpha) = 1 + \frac{1}{\alpha^2} - 2\frac{1}{\alpha} = \left(1 - \frac{1}{\alpha}\right)^2 > 0 \; \forall \alpha \neq 1$, so equation $h(x) = 0$ has the unique solution $\alpha = 1$.

Then using system (12) we have $\ln x_0 = -1$ and $\ln y_0 = -1$, hence $x_0 = e^{-1}, y_0 = e^{-1}$ and then $\lambda = e$. But this is contradiction and the Jacobian $\mathcal{J} \neq 0$ for $\lambda \neq e$.

Then functions $F_1(x, y, \lambda), F_2(x, y, \lambda)$ are differentiable infinity many times with respect to $(x, y, \lambda) \in (0, 1) \times (0, 1) \times (0, +\infty)$. Hence by implicit function theorem mappings $l(\lambda)$ and $ld(\lambda)$ are infinity differentiable with respect to $\lambda \neq e$. And hence functions $w(\lambda) = 1 - ld(\lambda)$ and $d(\lambda) = 1 - w(\lambda) - l(\lambda)$ are also infinity differentiable with respect to $\lambda \neq e$. ■

3.3 Continuity and differentiability, $\lambda = e$

Let's show that function $w(\lambda), l(\lambda), d(\lambda)$ are continuous but not differentiable in point $\lambda = e$.

Proposition 6. *Functions $w(\lambda), l(\lambda), d(\lambda)$ are continuous in $\lambda = e$.*

□ Show that $\lim_{\lambda \to e} l(\lambda) = l(e) = e^{-1}$. Suppose it's not true. Since $l(\lambda)$ is bounded function ($0 \leq l(\lambda) \leq 1$), there exists a sequence $\lambda_n \xrightarrow[n\to\infty]{} e$ for which $l(\lambda_n) \xrightarrow[n\to\infty]{} A \neq e^{-1}$. We know that for each λ the value of $l(\lambda)$ is the solution to $x = \exp\left(-\lambda e^{-\lambda x}\right)$. Hence $l(\lambda_n) = \exp\left(-\lambda_n e^{-\lambda_n l(\lambda_n)}\right)$ for all n. And passing to the limit

$$A = \lim_{n\to\infty} l(\lambda_n) = \lim_{n\to\infty} \exp\left(-\lambda_n e^{-\lambda_n l(\lambda_n)}\right) = \exp\left(-\lambda_0 e^{-\lambda_0 A}\right)$$

with $\lambda_0 = e$. But then we know that $A = e^{-1}$ and this is a contradiction.

Absolutely in the same way we may show, that function $ld(\lambda)$ is continuous in point $\lambda_0 = e$. And hence functions $w(\lambda), d(\lambda)$ are continuous in point $\lambda_0 = e$. ■

Theorem 7. *Functions $w(\lambda), l(\lambda), d(\lambda)$ are not differentiable in $\lambda = e$.*

□ To prove the theorem for $l(\lambda)$ it's sufficiently to show that $\lim_{\lambda \to e+0} l'(\lambda) = -\infty$.

For $\lambda > e$ we have that $l(x)$ is the least solution to $x = \exp\left(-\lambda e^{-\lambda x}\right)$. Hence:

$$\begin{gathered} l(\lambda) = \exp\left(-\lambda e^{-\lambda l(\lambda)}\right) \iff \ln l(\lambda) = \left(-\lambda e^{-\lambda l(\lambda)}\right) \iff \\ \ln\big(-\ln l(\lambda)\big) = \ln\lambda - \lambda l(\lambda) \iff \ln\big(-\ln l(\lambda)\big) - \ln\lambda + \lambda l(\lambda) = 0 \end{gathered} \tag{14}$$

and l is differentiable for $\lambda > e$. So we have:

$$\frac{l'(\lambda)}{l(\lambda)\ln l(\lambda)} - \frac{1}{\lambda} + l(\lambda) + \lambda l(\lambda) = 0 \iff l'(\lambda) = \frac{l(\lambda)\ln l(\lambda)(1 - \lambda l(\lambda))}{\lambda\big(\lambda l(\lambda)\ln l(\lambda) + 1\big)}, \quad \lambda > e. \tag{15}$$

Note that $\lim_{\lambda \to e+0} \dfrac{l(\lambda)\ln l(\lambda)}{\lambda} = -\dfrac{1}{e^2}$. Show that $\lim_{\lambda \to e+0} \dfrac{(1 - \lambda l(\lambda))}{\lambda l(\lambda)\ln l(\lambda) + 1} = +\infty$. We have

$$\frac{(1 - \lambda l(\lambda))}{\lambda l(\lambda)\ln l(\lambda) + 1} = \frac{1 + \ln(ld(\lambda))}{1 + \lambda\, ld(\lambda)\ln(ld(\lambda))}.$$

Let

$$\alpha_1 = \frac{1 + \ln(ld(\lambda))}{1 + \lambda\, ld(\lambda)\ln(ld(\lambda))} \quad \text{and} \quad \beta_1 = \frac{1 + \ln(ld(\lambda))}{1 + e\, ld(\lambda)\ln(ld(\lambda))}.$$

Firstly show that $\alpha_1(\lambda) > \beta_1(\lambda)$ for all $\lambda > e$.

Since $\lambda > e$ and $ld(\lambda)\ \ln ld(\lambda) < 0$, we have $1 + \lambda\, ld(\lambda)\ \ln ld(\lambda) < 1 + e\, ld(\lambda)\ \ln ld(\lambda)$.

Show that $1 + \lambda\, ld(\lambda) \ln ld(\lambda) > 0$. It's equivalent to $-\lambda\, ld(\lambda) \ln ld(\lambda) < 1 \iff \lambda^2\, ld(\lambda)\, l(\lambda) < 1$, which is true from lemma 3.2.

Show that $1 + \ln ld(\lambda) > 0$, it's the same that $ld(\lambda) > e^{-1}$. To show that it's sufficiently to prove that $ld(\lambda)$ increases for $\lambda \geq e$. Find $ld'(\lambda)$ from system:

$$\begin{cases} \dfrac{l'(\lambda)}{l(\lambda)} + ld(\lambda) + \lambda\, ld'(\lambda) = 0, \\ \dfrac{ld'(\lambda)}{ld(\lambda)} + l(\lambda) + \lambda\, l'(\lambda) = 0. \end{cases} \tag{16}$$

We obtain $ld'(\lambda) = \dfrac{l(\lambda)\, ld(\lambda)(\lambda ld(\lambda) - 1)}{1 - \lambda^2\, l(\lambda)\, ld(\lambda)}$. From lemma 3.2 the denominator is positive. The statement that the nominator is positive is equivalent to $ld(\lambda) > \frac{1}{\lambda}$. From system (9) we have $\frac{-\ln x_1}{x_3} = \lambda$. Since $\ln x_1 < \ln x_2 < \ln e^{-1} = -1$, we have $\frac{1}{x_3} < \lambda$ which is the same that $ld(\lambda) > \frac{1}{\lambda}$, so the nominator is positive.

Since the nominators of $\alpha_1(\lambda)$ and $\beta_1(\lambda)$ are equal and are positive, the denominator of $\alpha_1(\lambda)$ is less than the denominator of $\beta_1(\lambda)$, and both denominators are positive, so $\alpha_1(\lambda)$ is grater than $\beta_1(\lambda)$.

Show that $\lim\limits_{\lambda \to e+0} \beta_1(\lambda) = +\infty$. Since $ld'(\lambda) > 0$ for $\lambda > e$, we have $ld(\lambda) \xrightarrow[\lambda \to e+0]{} e^{-1} + 0$, so

$$\begin{aligned} \lim_{\lambda \to e+0} \frac{1 + \ln (ld(\lambda))}{1 + e\, ld(\lambda) \ln (ld(\lambda))} &= \lim_{y \to e^{-1}+0} \frac{1 + \ln y}{1 + e\, y\, \ln y} = \\ &= \lim_{y \to e^{-1}+0} \frac{1/y}{e + e \ln y} = +\infty. \end{aligned} \tag{17}$$

And thus $\lim\limits_{\lambda \to e+0} l'(\lambda) = -\infty$.

In very similar way we show that the function $ld(\lambda)$ is not differentiable in the point $\lambda = e$. It's sufficiently to show that $\lim\limits_{\lambda \to e+0} ld'(\lambda) = +\infty$.

For $\lambda > e$ we have that $ld(\lambda)$ is the solution to $x = \exp\left(-\lambda e^{-\lambda x}\right)$. Hence:

$$\begin{aligned} ld(\lambda) = \exp\left(-\lambda e^{-\lambda ld(\lambda)}\right) &\iff \ln ld(\lambda) = \left(-\lambda e^{-\lambda ld(\lambda)}\right) \iff \\ \ln\big(-\ln ld(\lambda)\big) = \ln \lambda - \lambda ld(\lambda) &\iff \ln\big(-\ln ld(\lambda)\big) - \ln \lambda + \lambda ld(\lambda) = 0 \end{aligned} \tag{18}$$

and ld is differentiable for $\lambda > e$. So we have:

$$\frac{ld'(\lambda)}{ld(\lambda) \ln ld(\lambda)} - \frac{1}{\lambda} + ld(\lambda) + \lambda ld(\lambda) = 0 \iff ld'(\lambda) = \frac{ld(\lambda) \ln ld(\lambda)(1 - \lambda ld(\lambda))}{\lambda\big(\lambda ld(\lambda) \ln ld(\lambda) + 1\big)}, \quad \lambda > e. \tag{19}$$

Note that $\lim\limits_{\lambda \to e+0} \dfrac{ld(\lambda) \ln ld(\lambda)}{\lambda} = -\dfrac{1}{e^2}$. Show that $\lim\limits_{\lambda \to e+0} \dfrac{(1 - \lambda ld(\lambda))}{\lambda ld(\lambda) \ln ld(\lambda) + 1} = -\infty$. We have

$$\frac{(1 - \lambda ld(\lambda))}{\lambda ld(\lambda) \ln ld(\lambda) + 1} = \frac{1 + \ln (l(\lambda))}{1 + \lambda\, l(\lambda) \ln (l(\lambda))}.$$

Let

$$\alpha_2 = \frac{1 + \ln (l(\lambda))}{1 + \lambda\, l(\lambda) \ln (l(\lambda))} \quad \text{and} \quad \beta_2 = \frac{1 + \ln (l(\lambda))}{1 + e\, l(\lambda) \ln (l(\lambda))}.$$

Firstly we show that $\alpha_2(\lambda) < \beta_2(\lambda)$ for all $\lambda > e$.

Since $\lambda > e$, $l(\lambda)\,\ln l(\lambda) < 0$ and $l(\lambda)\ln l(\lambda) = ld(\lambda)\ln ld(\lambda)$, so analogously to the previous proof we have $0 < 1 + \lambda\, l(\lambda)\,\ln l(\lambda) < 1 + e\, l(\lambda)\,\ln l(\lambda)$.

Show that $1 + \ln l(\lambda) < 0$. It's the same that $l(\lambda) < e^{-1}$, which is true from lemma 3.2.

Since the nominators of $\alpha_2(\lambda)$ and $\beta_2(\lambda)$ are equal and less than 0, the denominator of $\alpha_2(\lambda)$ is less than the denominator of $\beta_2(\lambda)$ and both denominators are positive, so $\alpha_2(\lambda)$ is less than $\beta_2(\lambda)$.

Show that $\lim\limits_{\lambda\to e+0}\beta_2(\lambda) = -\infty$. Firstly we need to prove that $l'(\lambda) < 0$ for $\lambda > e$. Find $l'(\lambda)$ from system

$$\begin{cases} \dfrac{l'(\lambda)}{l(\lambda)} + ld(\lambda) + \lambda\, ld'(\lambda) = 0, \\ \dfrac{ld'(\lambda)}{ld(\lambda)} + l(\lambda) + \lambda\, l'(\lambda) = 0. \end{cases} \tag{20}$$

We obtain $l'(\lambda) = \dfrac{ld(\lambda)\, l(\lambda)(\lambda l(\lambda) - 1)}{1 - \lambda^2\, l(\lambda)\, ld(\lambda)}$. As the denominator is positive for $\lambda > e$, so we have to verify condition $\lambda l(\lambda) - 1 < 0$. It is equivalently to $\ln ld(\lambda) > -1$. Since $\ln l(\lambda) < -1$, then by lemma 3.2 we obtain the desired statement.

Since $l'(\lambda) < 0$ for $\lambda > e$, we have $l(\lambda) \xrightarrow[\lambda\to e+0]{} e^{-1} - 0$. Consequently,

$$\lim_{\lambda\to e+0}\frac{1+\ln(l(\lambda))}{1+e\, l(\lambda)\ln(l(\lambda))} = \lim_{y\to e^{-1}-0}\frac{1+\ln y}{1+e\, y\,\ln y} = \tag{21}$$
$$= \lim_{y\to e^{-1}-0}\frac{1/y}{e+e\ln y} = -\infty.$$

And thus $\lim\limits_{\lambda\to e+0} ld'(\lambda) = +\infty$.

Since function $ld(\lambda)$ is not differentiable in $\lambda = e$, so function $w(\lambda) = 1 - ld(\lambda)$ is also not differentiable in $\lambda = e$. We have $\lim\limits_{\lambda\to e+0} d(\lambda) = 1 - \lim\limits_{\lambda\to e+0} w(\lambda) - \lim\limits_{\lambda\to e+0} l(\lambda) = +\infty$, hence $d(\lambda)$ is also not differentiable in $\lambda = e$. ■

Show that $\lim\limits_{\lambda\to e-0} l'(\lambda) = -\frac{1}{2e^2}$. For $\lambda < e$ we have that $l(\lambda)$ is solution to $\ln l(\lambda) + \lambda l(\lambda) = 0$. Hence $l'(\lambda) = -\frac{l^2(\lambda)}{1+\lambda l(\lambda)}$ for $(0, e)$. Thus $\lim\limits_{\lambda\to e-0} l'(\lambda) = -\frac{e^{-2}}{1+1} = -\frac{1}{2e^2}$.

We get $\lim\limits_{\lambda\to e-0} ld'(\lambda) = -\frac{1}{2e^2}$ and $\lim\limits_{\lambda\to e-0} d'(\lambda) = 1 - \lim\limits_{\lambda\to e-0} w'(\lambda) - \lim\limits_{\lambda\to e-0} l'(\lambda) = 1 - (-\frac{1}{2e^2}) - (1 + \frac{1}{2e^2}) = 0$.

Proposition 8. $w = \tilde{w}, d = \tilde{d}, l = \tilde{l}$.

□ Note that the iterative process for $\tilde{l}$ is the same $\tilde{l}_k = G(\tilde{l}_{k-2}), k \geq 2, k$ is even, and $\tilde{l}_0 = 0$. From the proof of theorem 4 it follows that $\tilde{l}_k$ converges to the least root of the equation $x = G(x)$. Hence $l = \tilde{l}$.

Since $l = \tilde{l}$, from the system (9), we have that $ld = \widetilde{ld}$, and so $w = \tilde{w}, d = \tilde{d}$. ■

We present the graphics of the functions $l(\lambda), d(\lambda), w(\lambda)$, obtained in Wolfram Mathematica:

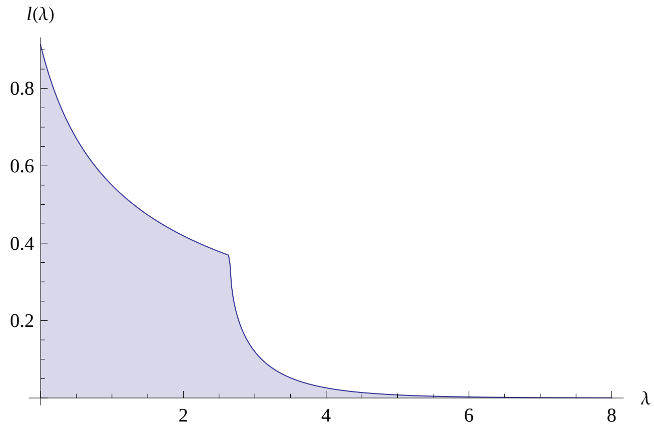

Figure 5: Probability that the first player loses in game $\mathcal{G}_1$

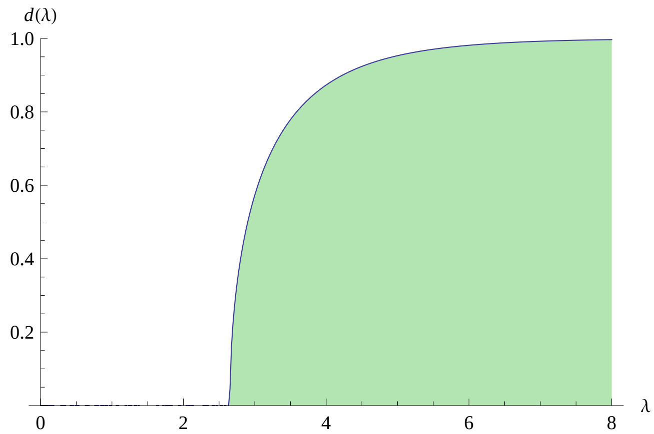

Figure 6: Probability that game $\mathcal{G}_1$ is drawing

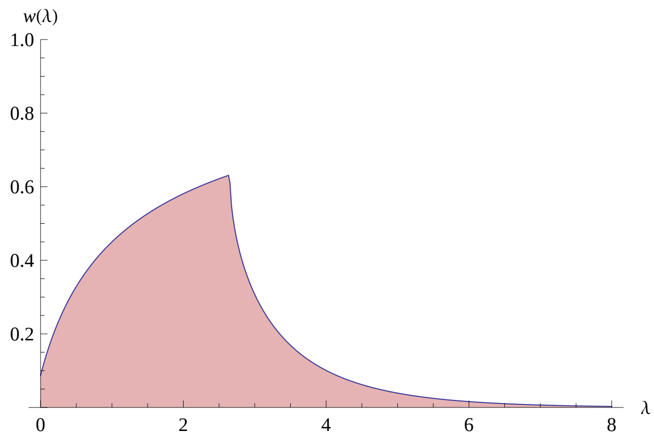

Figure 7: Probability that the first player wins in game $\mathcal{G}_1$

4 Geometric distribution

Consider the distribution defined as $\mu_i = qp^i$, where p, q are parameters, such that $p, q > 0, p + q = 1$. In this section we want to study properties of $p_{\text{ext}}, w(p), l(p), d(p)$ for this distribution.

Generating function $F(x)$ of this distribution is equal to:

$$F(x) = \sum_{i=0}^{\infty} qp^i x^i = \frac{q}{1-px} = \frac{1}{1+A(1-x)}, \tag{22}$$

where $A = \frac{p}{q}$. We get $F'(1) = A$.

Therefore, using results of the first section we have $p_{\text{ext}} = 1$ iff $p \leq q$.

Analogously to the Poisson distribution denote by $x = l = l(p)$ and $y = ld = l(p) + d(p)$. We have the system:

$$\begin{cases} y = F(1-x) \\ x = F(1-y) \end{cases} \iff \begin{cases} x = \frac{1}{1+Ay}, \\ y = \frac{1}{1+Ax}. \end{cases} \tag{23}$$

From it we have:

$$x = \frac{1}{1+A\left(\dfrac{1}{1+Ax}\right)} \iff \frac{1+Ax}{1+A+Ax} = x \iff \frac{Ax^2+x-1}{1+A+Ax} = 0.$$

Since $1+A+Ax > 0$, we have $x = \dfrac{-1 \pm \sqrt{1+4A}}{2A}$, and as $-1-\sqrt{1+4A} < 0$, so $x = \dfrac{-1+\sqrt{1+4A}}{2A}$. Substituting it to (23), we have

$$y = \frac{1}{1+A\left(\dfrac{-1+\sqrt{1+4A}}{2A}\right)} = \frac{2}{1+\sqrt{1+4A}} = \frac{2(1-\sqrt{1+4A})}{1-(4A+1)} = \frac{-1+\sqrt{1+4A}}{2A} = x.$$

Thus we have $l(p) = x = \dfrac{-1+\sqrt{1+4A}}{2A}$, $d(p) = y - x = 0$, $w(p) = 1 - y = \dfrac{2A+1-\sqrt{1+4A}}{2A}$. Since these functions are composition of elementary functions they are infinity differentiable with respect to $p \in (0,1)$.

5 Other directions

5.1 Strategy on tree

Redefine the games $\mathcal{G}_1$ and $\mathcal{G}_2$, in the following way. All rules stay the same, except the one, which says that in case of infinite tree the number of turns is limited with some constant M. It means that we truncate the tree on the M-th level. We say that the game is finished with the draw, if M moves were already made, but current vertex is not a leaf. Suppose we are given some tree (maybe infinite) and want to establish, whether it's winning, losing or drawing in new $\mathcal{G}_1, \mathcal{G}_2$.

We'll call vertices of the tree winning, drawing if the player have some sequence of turns that guarantees him win or only a draw, not depending on the turns of another player. Otherwise the vertex is losing for the one who moves from it.

In the game $\mathcal{G}_1$, if a vertex is a leaf, then it's losing, and in the game $\mathcal{G}_2$, if a vertex is a leaf, then it's winning. In both games, when a vertex is a leaf in truncated tree, and not a leaf in original tree, than it's drawing.

If vertex has some children, three cases are possible.

- If a vertex has losing children, then it's winning.
- If a vertex has no losing children, but have a drawing one, then it's drawing.
- And the vertex is losing, if it has only winning children.

Thus we can establish the type of the vertex in recursive way. Obviously the type of ancestor is the type of the tree.

5.2 Particular graphs

1. Let N be a natural number. Suppose we have a graph G which consists of N independent Galton-Watson trees $GW^i_\mu, i = \overline{1,N}$, with offspring distribution μ. Consider the modified games $\mathcal{G}_1, \mathcal{G}_2$ defined as the following (for simplicity we use the same notations $\mathcal{G}_1, \mathcal{G}_2$ for modified games). Let the first player chooses one of the trees GW^i_μ and plays the game $\mathcal{G}_1, \mathcal{G}_2$ on this tree. Set

$$\begin{aligned} w_1 &= \mathbb{P}\big(\text{first player wins the game } \mathcal{G}_1\big), \quad \tilde{w_1} = \mathbb{P}\big(\text{first player wins the game } \mathcal{G}_2\big) \\ l_1 &= \mathbb{P}\big(\text{first player loses the game } \mathcal{G}_1\big), \quad \tilde{l_1} = \mathbb{P}\big(\text{first player loses the game } \mathcal{G}_2\big) \\ d_1 &= \mathbb{P}\big(\text{draw in the game } \mathcal{G}_1\big), \quad \tilde{d_1} = \mathbb{P}\big(\text{draw in wins the game } \mathcal{G}_2\big) \end{aligned}$$

Then we obtain the relations

$$\begin{aligned} w_1 &= 1 - (l+d)^N, \quad l_1 = l^N, \quad d_1 = 1 - w_1 - l_1; \\ \tilde{w_1} &= 1 - (\tilde{l}+\tilde{d})^N, \quad \tilde{l_1} = \tilde{l}^N, \quad \tilde{d_1} = 1 - \tilde{w_1} - \tilde{l_1}. \end{aligned} \tag{24}$$

2. Let N be a natural number, $\xi_1, \dots \xi_N$ be independent random values, such that $\mathbb{P}(\xi_n = K_i) = \mu_i$; $n = \overline{1,N}$; $i = 0, 1, \dots$; $\mu_i \geq 0$; $\mu_0 = 0$; $\sum_{i=0}^\infty \mu_i = 1$; K_i is a complete graph with i vertices. We consider the games $\mathcal{G}_1, \mathcal{G}_2$, with condition that the first player chooses any graph K_i and the vertex in K_i.

Let probabilities $w_2, l_2, d_2, \tilde{w_2}, \tilde{l_2}, \tilde{d_2}$ be defined analogously to point 1.

We have that a complete graph with n vertices is winning in $\mathcal{G}_1$ iff n is even, and is winning in $\mathcal{G}_2$ iff n is odd. So the following relations hold

$$\begin{aligned} w_2 &= 1 - \left(\sum_{i=1}^\infty \mu_{2i-1}\right)^N, \quad l_2 = 1 - w_2, \quad d_2 = 0, \\ \tilde{w_2} &= 1 - \left(\sum_{i=0}^\infty \mu_{2i}\right)^N, \quad \tilde{l_2} = 1 - \tilde{w_2}, \quad \tilde{d_2} = 0. \end{aligned} \tag{25}$$

3. Let us consider independent random values ξ_1, ξ_2 taking the value i with probability μ_i, where $i = 0, 1, \dots$; $\mu_i \geq 0$; $\mu_0 = 0$; $\sum_{i=0}^\infty \mu_i = 1$. Consider a graph $G = K_{\xi_1, \xi_2}$ where $K_{i,j}$ is a complete bipartite graph with i vertices in the first partite and j vertices in the second one.

Consider games $\mathcal{G}_1, \mathcal{G}_2$ with condition that the first player chooses starting vertex. Define the probabilities $w_3, l_3, d_3, \tilde{w_3}, \tilde{l_3}, \tilde{d_3}$ as previously.

It's easy to see, that in the game $\mathcal{G}_1$ if the first player chooses a vertex in partite which is not greater than the other one, then he wins. So $w_3 = 1, l_3 = d_3 = 0$. In the game $\mathcal{G}_2$ the first player wins iff $\xi_1 \neq \xi_2$, consequently

$$\tilde{l_3} = \sum_{i=0}^{\infty} \mu_i^2, \quad \tilde{w_3} = 1 - \tilde{l_3}, \quad \tilde{d_3} = 0. \tag{26}$$

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