4^{th} International Tournament of Young Mathematicians

Problem 8: Discharged Batteries

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Abstract

This problem is about finding an optimal algorithm for a particular class of problems. Each problem is based on the following: there is a set of n batteries, which consists of c charged batteries and u uncharged ones. It is required to point out m charged batteries.

It is studied the case of $c = u$ in the first part. So, the optimal algorithm is constructed and its optimality is proved.

It is studied the case of $c \neq u$ in the second part. The algorithm is constructed for all cases. The optimality is proved for the case of $c < u \leq 2 \cdot c - 3, c > 2$.

The algorithm for solving the 2nd point of the problem and the lower bound of necessary 3-tries number are proposed in the third part.

Graph theory and combinatory are the main methods of the investigation.

Contents

1. Task #1

A racer has an electric car that needs two batteries to run. Unfortunately, the only reserve at his disposal is a set of $c+u$ *batteries (*$c \geq 2, u \geq 2, c = u$*) and the only thing he knows is that* c *of them are charged and the other* u *are uncharged. Call a 2-try the act of putting two batteries in the car and trying to make it run. It is required to find a strategy for the racer to apply in order to minimize the number of 2-tries needed to find a charged pair.*

Initial remarks:

- Call a n-try the act of putting n batteries in the car and trying to make it run.
- If we powered our car after at least one n-try, we found n charged batteries, so we assume we will get only negative results making a n-try ($n \geq 2$).
- The problem is represented in the form of a graph where a battery is a vertex. There will be an edge joining two vertices, if and only if we did a 2-try with this pair of batteries. This remark is applied to the points 1 and 2 of the report.

1.1. Lemma about a cycle of odd length

Lemma: The final graph for the solution of task #1 for $c \leq u$ should contain a cycle of odd length.

Proof:

Assumption: a final graph does not have an odd length cycle, the task has a solution.

If a graph contains no cycle of odd length, then the graph is bipartite. Consider the following cases:

1) Each part contains the same number of vertices. Then, we put the batteries of the same kind to the first part, remaining ones to the second part. So we can not point out any charged battery.
2) One of the parts contains more vertices than another. So, all of the charged batteries may be placed in the biggest part. In that part at least one battery is uncharged, because in the part the number of vertices is greater than at the other one. In this case we can not point out any charged batteries.

This is a contradiction, so the graph for the solution must contain a cycle of odd length.

1.2. The case $c = u = 2$

Separately, we consider the case $c = u = 2$, as it differs from the others.

It is necessary and sufficient to make 5 2-tries for solving the problem in the case of $c = u = 2$.

Sufficiency:

Since there are only two charged batteries, we can find them only if we checked all possible options of their location except the last one, which will be the desired, if we got negative results making the previous 2-tries.

Necessity:

Assumption: a final graph has no more than 4 edges, the task has a solution.

As the graph of the solution contains a cycle of odd length, then there are at least 3 edges.

Consider the following cases:

1) A graph has 3 edges. Any 3 vertices are connected into a cycle. Suppose the following location: one charged battery is located in the isolated vertex. Then the cycle has one charged battery and 2 uncharged ones. So, we can not point out 2 charged batteries.
2) A graph has 4 edges. Any 3 vertices are connected into a cycle. The rest edge connects the isolated vertex and any vertex of the cycle. Suppose the following location: the vertex of degree 1, contains a charged battery. The vertex of degree 3, contains an uncharged battery. Then one charged battery and one uncharged battery are located in the remaining vertices. So, we can not point out 2 charged batteries.

This is a contradiction. Thus, a graph of the solution should contain at least 5 edges.

1.3. The case $c = u, c > 2$

It is sufficient to make $c + 2$ tries for solving task #1 for $c = u$.

At first we choose any three vertices and connect them into a cycle. Then, obviously, there are at least 2 uncharged among them. Next, we connect the remaining $2 \cdot (c-2)$ batteries into pairs with $c-2$ 2-tries. Each pair contains at least one uncharged battery. So, the remaining battery is charged. Then, we connect it with any battery from any pair. Obviously, the second battery of that pair is charged. So we found 2 charged batteries.

1.4 The combinatorial method for lower estimates.

The combinatorial method for lower estimates is based on the following:

It is necessary to make $\left\lceil \frac{p-q}{q} \right\rceil, p = C_{u+c}^{c}, q = C_{c+u-\text{n}}^{c-\text{n}}, n \geq 2$ n-tries for solving task where $c+u$ is the number of batteries, c of them is charged, u of them is uncharged.

Proof:

There are $p = C_{u+c}^{c}$ different options of location c charged batteries. We exclude q options after making each n-try. Suppose, we pointed out n charged batteries, then there are no more than q locations. So we need to make at least $\left\lceil \frac{p-q}{q} \right\rceil$ n-tries.

1.5. The Base for future induction: $c = u = 3$

It is necessary and sufficient to make 5 2-tries for solving task #1 in the case of $c = u = 3$.

Sufficiency:

At first we choose any 3 vertices and join them into a cycle, so there are at least 2 uncharged batteries. Next, we connect the remaining vertices into a circuit, so it has at least 1 uncharged battery which is located in a vertex of degree 2. In this way the vertices of degree 1 contain charged batteries.

Necessity:

Assumption: a final graph has no more than 4 edges, the task has a solution.

Applying the combinatorial method for lower estimates we have $p = 20, q = 4$.

a) It is required to make at least 4 2-tries.

b) If we solved the problem with 4 2-tries, then each excluded option of locating 3 charged batteries should be excluded once.

As we have 4 edges and 6 vertices in the graph, then we have at least two adjacent edges. This is a contradiction to the statement b (edges connecting vertices A_1A_2, A_1A_3 both exclude option that charged batteries are located in this vertices). Thus a graph of the solution should contain at least 5 edges.

1.6. Terminology

We introduce the following definitions:

Isolated battery — a battery, corresponding to an isolated vertex of a final graph.

Mutually free batteries — batteries, which correspond to the vertices of the graph and are not connected with each other.

Satisfactory edge — an edge, which is incident to a vertex with an uncharged battery.

1.7. The introduction of additional information method

The introduction of additional information method is based on the following statement: if the player can solve the given task then, if we suggest an additional information, hence he can solve it.

1.8. The proof of algorithm optimality for solving task #1.3

The base of induction is the task for $c = u = 3$ (see section #1.5). We proceed by induction: suppose we proved the previous statement for $c = u = k-1$, then we will prove it for $c = u = k$. To prove the induction step we apply the method of isolated batteries. It is based on the following: at first, we prove that the graph of the solution can not contain more than 2 isolated vertices. A further decision will be reduced to simple sorting.

Suppose the task for $c = u$ has been solved with $c+1$ 2-tries. Let us consider the graph of the solution. Suppose there are at least 3 isolated batteries. If the graph contains at least $c+1$ components, there are always $c+1$ mutually free batteries such that if they contain c charged and one discharged batteries we can not find any charged battery. Otherwise, the graph contains no more than c components. The graph contains $c+1$ edges, and $2 \cdot c$ vertices. Then there are at least $2 \cdot c - (c+1) = c-1$ components. As there is a cycle, the graph contains at least c components (At first we choose any $c+1$ vertices and join them into a circuit with c edges. The rest edge can connect any 2 vertices of the circuit. In this case the graph has the lowest number of components which equals c). So, the graph contains c components.

Notice that each component is a complete graph. Indeed, suppose that two vertices of the same component are not connected. Then we choose one vertex from every remaining component. Then together with these two vertices they form $c+1$ mutually free batteries, and in this case, as was shown above, the task is not solved.

Suppose that there is a component with two vertices. Then we suggest to the player that this component contains one charged and one uncharged battery. So, according to the introduction of additional information method player can solve the task. As these 2 vertices are not connected with other vertices of the graph, then applying induction it can be dropped, and the task is reduced to the case $c = u = k-1$. Otherwise, every component contains either one or at least 3 vertices.

As every component with at least 3 vertices is a complete graph, then it has at least the same number of vertices and edges. Then all components with 3 and more vertices contain no more than $c+1$ vertices, because the number of edges is no more than $c+1$ in the whole graph. So, there are at least $c-1$ isolated vertices. As there are c components, then the remaining vertices form a single component. A complete graph has $\frac{c \cdot (c-1)}{2}$ edges which is more than $c+1$ for $c > 2$. This is a contradiction. So, the graph contains no more than two isolated batteries.

Let us consider different options of a graph of the solution for the task #1.3 with no more than 2 isolated vertices. Suppose that the graph has a cycle with 5 vertices. Then remaining $2 \cdot c - 8$ vertices are uniquely connected into pairs with $c-4$ edges. The 3 isolated vertices remain, which is a contradiction. It is obvious that if the number of the vertices in a cycle increases then the number of the isolated vertices in the graph of the solution increases as well. Thus the cycle should consist of 3 vertices. The remaining $2 \cdot c - 6$ vertices are connected into pairs with $c-3$ edges. The rest edge can connect the following vertices:

1) any 2 vertices from the different pairs;
2) any vertex of a pair and the isolated vertex;
3) a vertex of the cycle and any other vertex.

Any option of the graph allows us to choose $c+1$ mutually free batteries. This is a contradiction. So, it is necessary and sufficient to make $c+2$ 2-tries for solving the task #1 in the case of $c = u, c > 2$.

2. Task #2

It is required to investigate the case when there are $c > 2$ charged batteries, $u > 0$ uncharged ones and $c \neq u$.

2.1. The principle of $u-1$ uncharged batteries

Suppose that we managed to put $u-1$ uncharged batteries in such a way that all edges became satisfactory so that every edge is incident to a vertex with an uncharged battery. Then we can not find 2 charged batteries.

Proof:

If we were able to put $u-1$ batteries, as was shown above, obviously, the remaining $c+1$ batteries can be placed in random vertices. In particular, the remaining uncharged battery can be placed into any vertex from $c+1$ remaining ones, and thus we can not find any charged battery.

2.2. The case $c > u + 1$

It is necessary and sufficient to make u 2-tries for solving task #2 for $c > u + 1$.

Sufficiency:

At first we connect $2 \cdot u$ vertices into pairs. Each pair has one uncharged battery. Then the remaining $c - u$ batteries are charged.

Necessity:

Assumption: a final graph has no more than $u - 1$ edges, the task has a solution.

Suppose the task was solved for $u - 1$ 2-tries. Let the batteries place in the following manner: at each step we will put one of $u - 1$ uncharged batteries into a vertex of the highest degree. After that all the edges incident to this vertex should not be considered while counting the degree of the remaining vertices. We can always do it, because the graph of the solution has $u - 1$ edges. The rest uncharged battery and c charged batteries can be placed into any vertex from $c + 1$ remaining ones. According to the principle of $u - 1$ uncharged batteries we can not find any charged battery. This is a contradiction, so a graph of the solution should contain at least u edges.

2.3. The case $c = u + 1$

It is necessary and sufficient to make $u + 1$ 2-tries to solve the task #2 for $c = u + 1$.

Sufficiency:

Let $2 \cdot u$ vertices connect into pairs with u edges. Such a pair has one uncharged battery. The remaining battery is uniquely charged. Then this battery is connected to a battery from any pair. The other battery of the pair is charged.

Necessity:

Assumption: a final graph has no more than u edges, the task has a solution.

Consider the following 2 cases:

1) There is a vertex of degree at least 2. We put one uncharged battery into this vertex. Then no more than $u - 2$ unsatisfactory edges remain (here an unsatisfactory edge stands for any edge except for a satisfactory one). At each step we will put one of $u - 2$ uncharged batteries into a vertex which has the highest degree. After that all the edges incident to this vertex should not be considered while counting the degree of the remaining vertices. We can always do it, because the graph of the solution has no more than $u - 2$ unsatisfactory edges. The rest uncharged battery and c charged batteries can be placed into any vertex from $c + 1$ remaining ones. According to the principle of $u - 1$ uncharged batteries we can not find any charged battery.

2) There are all vertices of degree no more than 1. Without loss of generality, the batteries are connected in the following manner: $\#1 - \#2, \#3 - \#4, \ldots, \#2 \cdot u - 1 - \#2 \cdot u$. It is obvious that the battery $\#2 \cdot u + 1$ is charged, as every pair has both charged and uncharged battery. According to the principle of $u - 1$ uncharged batteries we can not find any charged battery.

So this is a contradiction and a graph of the solution should contain at least $u + 1$ edges.

2.4 The case $u > c, c = 3$

It is sufficient to make $\frac{(u-1)\cdot(u-2)}{2} + 5$ 2-tries for solving the task #2 when $u > c, c = 3$.

Sufficiency:

Let any $u - 1$ vertices connect into the complete graph. Then there are at least $u - 2$ uncharged batteries. So we can obtain the task when $c = u = 2$ which requires to make 5 2-tries for solving it.

2.5. The case $c < u < 2 \cdot c - 3, c > 3$

It is necessary and sufficient to make $c+2\cdot k+2$ 2-tries for solving task #2 for $c<u<2\cdot c-3$, $k=u-c, c>3$.

Sufficiency:

At the beginning we choose k disjoint sets of 3 vertices and join the set vertices into cycles. Every cycle has at least 2 uncharged batteries. Then the task of the remaining batteries is reduced to the task with the same number of charged and uncharged batteries, which has $2\cdot(c-k)$ batteries and requires making $c-k+2$ 2-tries for solving it.

Necessity:

Assumption: a final graph has no more than $c+2\cdot k+1$ edges, the task has a solution.

Suppose we showed the locations of k uncharged batteries. According to the introduction of additional information method player can solve the task. As these vertices contain uncharged batteries and all edges correspond to negative 2-tries we can remove this vertex and its incident edges from the graph and get the equivalent task. At each step we will put one of k uncharged batteries into a vertex which has the highest degree and then remove this vertex and its incident edges from the graph. Since

If we choose a set of k batteries as indicated above, then all the batteries of such a set may be uncharged. Consider the graph obtained after applying the introduction of additional information method.

Consider the following 2 cases:

1) We removed a vertex of degree no more than 1 at least once. As we removed the vertex of the highest degree on every step, every vertex in the remaining graph has degree no more than 1. So the graph has no more than c edges. After that at each step we will put one of c uncharged batteries into a vertex which has the highest degree. Then we remove this vertex and its incident edges from the graph.

2) We always removed the vertices of degree at least 2. Thus, the obtained graph has no more than $c+1$ edges and c both charged and uncharged batteries. So, there is at least one vertex of degree at least 2, where we can put an uncharged battery. We can remove this vertex and its incident edges from the graph and get the graph with no more than $c-1$ edges. After that at each step we will put one of $c-1$ uncharged batteries into a vertex which has the highest degree. Then we remove this vertex and its incident edges from the graph.

So we can put all uncharged batteries in such a way that all the edges will become satisfactory and indicated k batteries will be uncharged.

Now we will show that the task obtained by applying the introduction of additional information method can not be solved, in other words we can not find 2 charged batteries.

Suppose we removed a vertex of degree no more than 1. As we removed the vertex of the highest degree at every step, then in the obtained graph every vertex has degree no more than 1. So, the graph has no more than c edges. This task is unsolvable (see section #1).

Suppose we always removed a vertex of degree at least 2. So the obtained graph has c charged and $u-k=c$ uncharged and no more than $c+2\cdot \mathrm{k}+1-2\cdot \mathrm{k}=c+1$ edges. But this task is also unsolvable (see section #1).

This is a contradiction, so a graph of the solution should contain at least $c+2\cdot k+2$ edges.

2.6. The case $m\cdot c-3\cdot(m-1)<u\leq(m+1)\cdot c-3\cdot m$ where $c>3, m\in \mathrm{N}$

Let $n_c=\left\lceil \frac{u-c}{m} \right\rceil$. It is sufficient to make $\frac{n_c\cdot(m+2)\cdot(m+1)}{2}+2\cdot(c-n_c)+2$ 2-tries for solving this task.

Sufficiency:

At the beginning we choose n_c disjoint sets of $m+2$ vertices and join the set vertices into complete graphs. It requires to make $\frac{n_c\cdot(m+2)\cdot(m+1)}{2}$ 2-tries. Then the task will be reduced to the task with $c'\geq c-n_c$ charged batteries and $u'\leq u-n_c\cdot(m+2)$ discharged batteries. In this case $c'\geq u' \Leftarrow c-n_c\geq u-n_c(m+1)\Leftarrow n_c >=\frac{u-c}{m}$. The number of 2-tries depends on the following 2 cases:

1) $c'+u'=2\cdot k$. It is sufficient to make $k+2$ 2-tries for solving the task (see section #1.2 or #1.3).

At the beginning we apply the algorithm for solving the task #2 in the case of $c'=u'$, for which it is sufficient to make $c'+2$ 2-tries. Let us show that this algorithm also solves the task #2 in the case $c'>u'$.

Suppose, the task #2 is solved in the case of $c'=u'=k$. It means that we can point out 2 vertices A and B, which contain 2 charged batteries. So, if we put uncharged battery in the vertex A, then we can not put the remaining $u'-1$ uncharged batteries in such a way all edges become satisfactory. The same is true for the vertex B.

Suppose, we apply the same algorithm for solving the task #2 in the case of $c'=k+l>u'=k-l$. Let us show that the following statement is true: the vertex A contains the charged battery. Indeed, let the charged battery is located in the vertex A. Then, we can put the remaining $u'-1=k-l-1$ uncharged batteries in such a way all edges become satisfactory. But we can not make it, as was shown above, even $k-1$ uncharged batteries is not enough. The same is true for the vertex B. Thus, we can point out 2 charged batteries.

This is a contradiction, so it is sufficient to make $k+2$ 2-tries for solving the task #2 in the case of $c'+u'=2\cdot k$

2) $c'+u'=2\cdot k+1$. It is sufficient to make $k'+1$ 2-tries for solving the task (see section #2.3).

Similarly with the previous case it is sufficient to make $k'+1$ 2-tries for solving the task #2 in the case of $c'+u'=2\cdot k+1$.

2.7. Another algorithm for solving the task #2.6

We also developed the second algorithm for solving the task #2.6 based on recursion. As we suppose it is more optimized than the first one.

The algorithm is based on the following actions:

Let $n_c=\left\lceil\frac{u-m\cdot c+3\cdot(m-1)}{m}\right\rceil$. At first, we choose n_c disjoint sets of $m+2$ vertices and join the set vertices into complete graphs. It is required to make $\frac{n_c\cdot(m+2)\cdot(m+1)}{2}$ 2-tries. Then we remove these graphs and the task will be reduced to the task with $c'\geq c-n_c$ charged batteries and $u'\leq u-n_c\cdot(m+1)$ discharged batteries. In this case for $m'=m-1$ we have:

$$(m'+1)\cdot c'-3\cdot m'\geq u' \Leftarrow (m'+1)\cdot(c-n_c)-3\cdot m'\geq u-n_c(m+1)\Leftarrow n_c >=\frac{u-m\cdot c+3\cdot(m-1)}{m}.$$

This way, we reduce initial configuration to the case when $\tilde{m}\leq m-1$.

We will apply the same algorithm further, until we get the task when $c\geq u$.

3. Task #3

Now, suppose that the car is bigger and it needs three charged batteries instead. Suppose also that the racer has only eight batteries, but still a symmetric distribution between charged and uncharged. It is required to find a strategy to minimize the number of 3-tries.

3.1. The case $c=u=4$

It is sufficient to make 21 3-tries for solving task #3 in the case of $c=u=4$.

Sufficiency:

At first step we divide batteries into 2 sets of 4 batteries each. Then we make all possible 3-tries in each set. So, each set has 2 charged and 2 uncharged batteries. Then we choose any battery from the first set and make all possible 3-tries with any pair of batteries from the second set. Then we remove this battery as it is uncharged. Then we apply the same step for another set. We will continue, until 2 batteries remain in each set and for sure these batteries are charged.

Necessity:

The combinatorial method for lower estimates says that we need to make at least 13 3-tries.