4th International Tournament of Young Mathematicians

Team: Belarus

Problem 7. An experiment

Abstract

We show that this is a linear operator and we found and proof some theorems about it kernel and image. We investigate the reverse operation. Also we found criteria for 1-balanced($\sum_{i}^{n}(\alpha_{0;i}\alpha_{0;i+1}) = \frac{n}{4}$ or the number of G-set is equal $\frac{n}{2}$), 2-balansed, $\sum_{i=1}^{n}\alpha_{0;i+2} * \alpha_{0;i} = \frac{n}{4}$ or the number of G-sets with a power of 1 is $\frac{n}{4}$) and 3-balansed sequences.

We paid great attention to the state graph of this operator. We found and proved the following propositions:

1. “Null” tree is a full binary tree.

2. Each tree with the root on each point of each cycle of the graph G isomorphic to the "null" tree.

3. In some cases we investigate the number of components.

We found and proved some theorems, connected with circles in our graph.

Introduction

Consider the operator $\varphi : GF(2)^n \to GF(2)^n$ over the field $GF(2)$ of two elements, such that $\varphi(\alpha_1,\ldots,\alpha_n) = (\alpha_1+\alpha_2,\ldots,\alpha_n+\alpha_1)$.

Note that this is linear operator:

- $\varphi(\vec{\alpha}+\vec{\beta}) = \varphi(\vec{\alpha})+\varphi(\vec{\beta})$
- $\varphi(\lambda\vec{\alpha}) = \lambda\varphi(\vec{\alpha})$

The i-th element after k-th operation is called $\alpha_{k;i}$, $\alpha_{k;i} \equiv \alpha_{k-1;i} + \alpha_{k-1;i+1}$ (mod 2)

The sequence balanced after 1,…,k operation is called *k-balanced sequence.*

Sum of all possible products of i elements from k elements is noted σ_i^k

The set of k elements (all elements go one by one) is called *under-sequence.*

The set of k elements (all element go one by one) and for any element $x_i = x_{i+1}$ (we think that our multitude is circle) is called *G-set*.

G-set with number of elements to equal k is noted *(k).*

Binary Pascal's triangle is a Pascal's triangle modulo 2.

Stick together is operation with adjacent G-sets, after this operation adjacent G-set become one G-set and sum of powers adjacent G-sets is equal power new G-set.

1. Reverse operation.

Proposition 1. If $\varphi(\vec{\alpha}) = \vec{b} \Rightarrow \varphi(\vec{\alpha}+e) = \vec{b}, where\ e = (1,...,1)$.

It's true, because $\varphi(\vec{\alpha}+e) = \varphi(\vec{\alpha})+\varphi(e) = \vec{b}+\vec{0}$.

We have, that $\varphi(\alpha_1,\ldots,\alpha_n) = (\alpha_1+\alpha_2,\ldots,\alpha_n+\alpha_1) = (\beta_1,\ldots,\beta_n)$, then we have,

$$\alpha_3+\alpha_1 \equiv \beta_2-\beta_1 \equiv \beta_2+\beta_1,\ \alpha_4+\alpha_1 \equiv \beta_1+\beta_2+\beta_3,\ldots\ ,\ \alpha_n+\alpha_1 = \sum_{i=1}^{n-1}\beta_i .$$

Lemma 1. About the image of operator

$$\beta(\beta_1,\ldots,\beta_n) \in \operatorname{Im}\varphi \Leftrightarrow \sum_{i=1}^{n-1}\beta_i \equiv \beta_n$$

Proof.

$\Rightarrow$We know, that $\alpha_n+\alpha_1 \equiv \sum_{i=1}^{n-1}\beta_i$ and $\alpha_n+\alpha_1 \equiv \beta_n$

$\Leftarrow$using proposition 1 we WLOG can suppose, that $\alpha_1 = 0$, then we can recover all elements of α using $\alpha_j \equiv \sum_{i=1}^{j-1}\beta_i$. And this α satisfied $\varphi(0,\ldots,\alpha_n) = (\beta_1,\ldots,\beta_n)$, because

$$\alpha_j+\alpha_{j+1} \equiv \sum_{i=1}^{j-1}\beta_i + \sum_{i=1}^{j}\beta_i \equiv \beta_j \text{ and } \alpha_n+\alpha_1 \equiv \sum_{i=1}^{n-1}\beta_i \equiv \beta_n .$$

2. Algebraic method

Theorem 2.1. *The sequence* α *is 1-balansed if and only if* $\sum_{i}^{n}\left(\alpha_{0;i}\alpha_{0;i+1}\right)=\frac{n}{4}$ *and* $\sum_{i=1}^{n}\alpha_{0;i}=\frac{n}{2}$

Proof.

Let sequence $\alpha_{0;1};\alpha_{0;2};\ldots;\alpha_{0;n}$ be 1-balanced sequence. So we have that $\sum_{i=1}^{n}\alpha_{0;i}=$ n/2 and $\sum_{i=1}^{n}\alpha_{1;i}=n/2$.

$$\sum_{i=1}^{n}\alpha_{1;i}=\sum_{i=1}^{n}\left(\alpha_{0;i}+\alpha_{0;i+1}-2\alpha_{0;i+1}*\alpha_{0;i}\right)=2\sum_{i=1}^{n}\alpha_{0;i}-2\sum_{i=1}^{n}\alpha_{0;i+1}\alpha_{0;i}=$$

$$=n-2\sum_{i=1}^{n}\alpha_{0;i+1}\alpha_{0;i}$$

Note that $\alpha_{k;i}=\alpha_{k-1;i}+\alpha_{k-1;i+1}-2\alpha_{k-1;i}\alpha_{k-1;i+1}$. So we have $\sum_{i}^{n}(\alpha_{0;i}\alpha_{0;i+1})=\frac{n}{4}$. QED $\frac{n}{4}$ is integer, so we have *n* divide by 4.

Theorem 2.2

There is no 1-balansed *n*-tuple if *n*=4*k*+2. And all balanced sequences not in the image of operator.

Proof. If the tuple $(\beta_1,\ldots,\beta_n)$ balanced, then exactly half of its elements are zeros. Consider the sum $\sum_{i=1}^{n}\beta_i$ this sum is equal to one, because exactly 2k+1 elements is zeros. But according the lemma 1 this elements are not in the image of operator. End doesn't exist α, such that $\varphi(\alpha)=\beta$.

2.3. *A necessary and sufficient condition for2-balanced sequence is* $\sum_{i=1}^{n}\alpha_{0;i+2}*\alpha_{0;i}=\frac{n}{4}$ *and condition for 1-balanced sequence.*

Proof:

We have:

$$\alpha_{2;i}=(\alpha_{1;i}-\alpha_{1;i+1})^2=((\alpha_{0;i}-\alpha_{0;i+1})^2-(\alpha_{0;i+1}-\alpha_{0;i+2})^2)^2=$$

$$=((\alpha_{0;i}-\alpha_{0;i+1}-\alpha_{0;i+1}+\alpha_{0;i+2})(\alpha_{0;i}-\alpha_{0;i+1}+\alpha_{0;i+1}-\alpha_{0;i+2}))^2=$$

$$=((\alpha_{0;i}-\alpha_{0;i+2})(\alpha_{0;i}-2*\alpha_{0;i+1}+\alpha_{0;i+2}))^2$$

Let prove that $\alpha_{2;i}=(\alpha_{0;i}-\alpha_{0;i+2})^2$. If $\alpha_{0;i}=\alpha_{0;i+2}$ then our supposition is true. If $\alpha_{0;i}\neq\alpha_{0;i+2}$ we have that

$$\alpha_{0;i}+\alpha_{0;i+2}=1\Rightarrow(\alpha_{0;i}-2*\alpha_{0;i+1}+\alpha_{0;i+2})^2=1\Rightarrow\alpha_{2;i}=(\alpha_{0;i}-\alpha_{0;i+2})^2$$

So $\alpha_{2;i}=(\alpha_{0;i}-\alpha_{0;i+2})^2$ then we have that $\sum_{i=1}^{n}\alpha_{0;i+2}*\alpha_{0;i}=\frac{n}{4}$
(analogically with 1-balanced sequence). QED.

2.4 *If* $a\equiv b_1+b_2+...+b_y$ *(mod 2) then* $a=\sigma_1^y-2*\sigma_2^y+4\sigma_3^y-...2^{y-1}\sigma_y^y$ *(*b_i *is integer)*

Let we have m element is equal 1 in our condition $b_1;b_2;...;b_y$. Then we have

$$\sigma_1^y-2*\sigma_2^y+4\sigma_3^y-...2^{y-1}\sigma_y^y=C_m^1-2C_m^2+...2^{m-1}C_m^m=\frac{2C_m^1-4C_m^2+...2^mC_m^m}{2}=$$

$$\frac{1-1+2C_m^1-4C_m^2+...2^mC_m^m}{2}=\frac{1-(1-2)^m}{2}$$

If m is odd, then $\frac{1-(1-2)^m}{2}$ =1=a. If m is even, then $\frac{1-(1-2)^m}{2}$ =0=a

2.5. $\alpha_{k;i}\equiv C_{k-1}^0\alpha_{0;i}+C_{k-1}^1\alpha_{k;i+1}+...+C_{k-1}^{k-1}\alpha_{k;i+k-1}$ *(mod 2)*

We prove this fact with the binary Pascal's triangle (let see the illustration).It's easy to see, that this is Pascal's triangle because $x_{k;i}\equiv x_{k-1;i}+x_{k-1;i+1}$.

$\alpha_{k;1}\equiv c_1\,\alpha_{0;1}+c_2\,\alpha_{0;2}+...+c_{k+1}\,\alpha_{0;k+1}$,(where α_i is number located at the intersection of x_i and k)																	
k	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	00	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
6	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0
7	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
8	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
9	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
10	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0
11	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0
12	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
13	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0
14	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

For example $\alpha_{2^l;i}\equiv\alpha_{0;i}+\alpha_{0;2^l+i}$ or $\alpha_{2^l-1;i}\equiv\sum_{k=i}^{i+2^i}\alpha_{0;k}$

3.Graph

Consider the set F_m={0; 1; 2; 3 …;m-1} of residues modulo m. And we see sum by module m. We say that our sequence is balanced if sum of element of sequence is equal $\frac{n}{m}$.

Reverse operation

Let sequence $\beta_1;\beta_2;...;\beta_n$ after 1 operation λ will be sequence $\lambda_1;\lambda_2;...;\lambda_n$. Suppose β_1=0 then

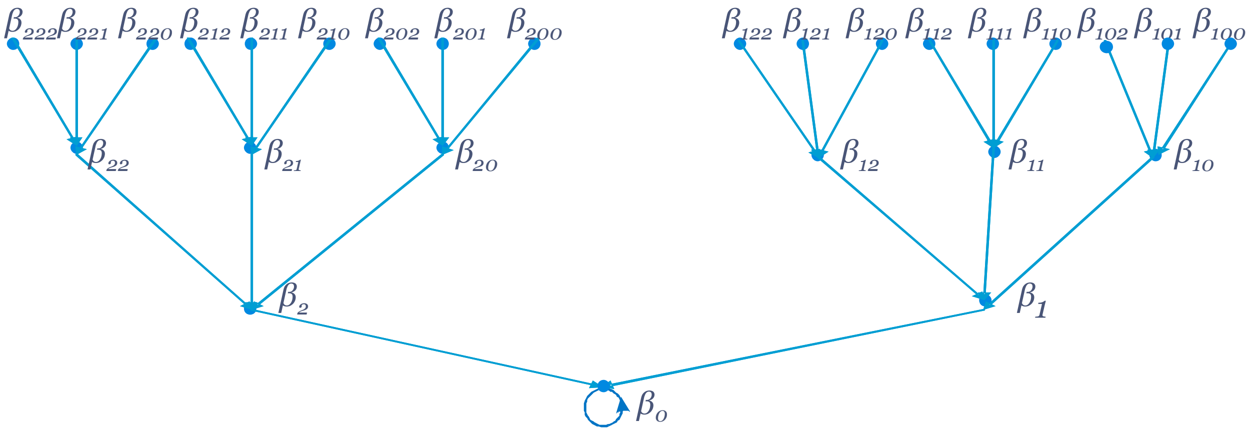
$\beta_1+\beta_2\equiv\lambda_1\Rightarrow\beta_2\equiv\lambda_1-\beta_1;\lambda_2\equiv\beta_2+\beta_3\Rightarrow\beta_3\equiv\lambda_2-\beta_2\equiv\lambda_2-\lambda_1+\beta_1$. So we have $\beta_m=\lambda_{m-1}-\lambda_{m-2}+\lambda_{m-3}-...\pm\beta_1$. So we can find all sequence $\beta_1;\beta_2;...;\beta_n$.

The vertex graph is a sequence. The lines are operations. Each vertex has 1 or m+1 lines (this is simply to see from reverse operation). Those each connected component of a graph has only 1 cycle. At each vertex of the cycle will be constructed trees.

m-th floor is the set of all vertices and their distance to root is m

Vertex located on the m floor, and includes $\beta_{i_1 i_2 ... i_{m-1}}$ is called $\beta_{i_1 i_2 ... i_m}$.

e_n = (0,0,…,0) is called "zero trees"



3.1 *Let we have:* $\beta,\beta_1,\beta_{10},...,\beta_{\underbrace{100..0}_{m-1}}$ *then* $\beta_{i_1 i_2 ... i_m}=i_1\beta_{\underbrace{100..0}_{m-1}}+i_2\beta_{\underbrace{100..0}_{m-2}}+...+i_m\beta_1$.

Let use induction:

The base m=1:

$\beta_{i_1}=i_1\cdot\beta_1$ *for* $\forall i_1\in\{0,1,p-1\}$, really $\beta_{i_1}=\varphi(i_1\cdot\beta_1)=i_1\cdot\varphi(\beta_1)=i_1\cdot\beta=\beta$ .It is simply to see that $\beta_0,\beta_1,\beta_2,...,\beta_{p-1}$ is different vertex.

Let preposition is true for m = *l-1*, so we have: $\beta_{i_1 i_2 ... i_{l-1}}=i_1\beta_{\underbrace{100..0}_{l-2}}+i_2\beta_{\underbrace{100..0}_{l-3}}+...+i_l\beta_1$.

If we prove that $\varphi(\chi)=\beta_{i_1 i_2 ... i_l}$, *where* $\chi=i_1\beta_{\underbrace{100..0}_{l-1}}+i_2\beta_{\underbrace{100..0}_{l-2}}+...+i_l\beta_0$ then we using the construction $\beta_{i_1 i_2 ... i_l}$, show that $\beta_{i_1 i_2 ... i_l}=\chi$.

So φ is linear we have:

$$\varphi(\chi)=\varphi(i_1\beta_{\underbrace{100..0}_{l-1}}+i_2\beta_{\underbrace{100..0}_{l-2}}+...+i_l\beta_0)=i_1\varphi(\beta_{\underbrace{100..0}_{l-1}})+i_2\varphi(\beta_{\underbrace{100..0}_{l-2}})+...+i_l\varphi(\beta_0)=$$

$$i_1\beta_{\underbrace{100..0}_{l-2}}+i_2\beta_{\underbrace{100..0}_{l-3}}+...+i_{l-1}\beta_1+i_l\beta_0=i_1\beta_{\underbrace{100..0}_{l-2}}+i_2\beta_{\underbrace{100..0}_{l-3}}+...+i_{l-1}\beta_1=\beta_{i_1 i_2 ... i_{l-1}}$$

QED

3.2.

Let see the tree with the root is equal e_n. *All elements* $\beta_{i_1 i_2 \ldots i_m}$ *is different (for any* $i_1; i_2; \ldots; i_m$*).*

Let we have 2 equal elements: $\beta_{x_1 x_2 \cdots x_m} = \beta_{y_1 y_2 \cdots y_m}$. So we have:

$$x_1\beta_{\underbrace{100\ldots0}_{m-1}} + x_2\beta_{\underbrace{100\ldots0}_{m-2}} + \ldots + x_m\beta_1 = y_1\beta_{\underbrace{100\ldots0}_{m-1}} + y_2\beta_{\underbrace{100\ldots0}_{m-2}} + \ldots + y_m\beta_1 \Rightarrow$$

$(x_1 - y_1)\beta_{\underbrace{100\ldots0}_{m-1}} + (x_2 - y_2)\beta_{\underbrace{100\ldots0}_{m-2}} + \ldots + (x_m - y_m)\beta_1 = 0$. Let k be a minimal number such that $x_k \neq y_k$ so we have:

$$(x_k - y_k)\beta_{\underbrace{100\ldots0}_{k-1}} + (x_{k+1} - y_{k+1})\beta_{\underbrace{100\ldots0}_{m-k-1}} + \ldots + (x_m - y_m)\beta_1 = 0$$

$$\Rightarrow \varphi^{m-k}((x_k - y_k)\beta_{\underbrace{100\ldots0}_{k-1}} + (x_{k+1} - y_{k+1})\beta_{\underbrace{100\ldots0}_{m-k-1}} + \ldots + (x_m - y_m)\beta_1) = \varphi^{m-k}(0) = 0 \Rightarrow$$

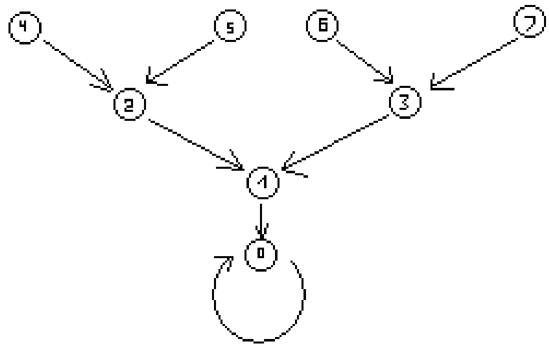
$$(x_k - y_k)\varphi^{m-k}\beta_{\underbrace{100\ldots0}_{k-1}} + (x_{k+1} - y_{k+1})\varphi^{m-k}\beta_{\underbrace{100\ldots0}_{m-k-1}} + \ldots + (x_m - y_m)\varphi^{m-k}\beta_1 = 0 \Rightarrow$$

$$(x_k - y_k)\varphi^{m-k}\beta_{\underbrace{100\ldots0}_{k-1}} = 0$$

But this is impossible, since $x_k \neq y_k$. So we haven't 2 equal elements. QED.

3.3

Each tree with the root on each point of each cycle of the graph G_ψ isomorphic to the "null" tree.



Proof:

For all sequences k and l, are in one floor of a tree, (here we have *l* is in the cycle and *k* on the tree),the following condition holds:

$$\varphi^n(k) = \varphi^n(l)$$
$$\varphi^{n-1}(k) \neq \varphi^{n-1}(l)$$

The following equality is true:

$k - l = G_n$, where G_n — one of the sequences of "null" tree on the *n* floor (the sum in the field Z_2).

Using this ratio can be completed to any tree is a tree isomorphic to "null".

In next theorem m=2.

3.4. *If sequence have* 2^l *element then graph will have one connected component.*

If sequence have 2^l element, then after 2^l-1 operation sequence will equal e_n or e (if sum of element is even then sequence will be e_n else sum is add sequence will be e) and after 2^l operation sequence will equal e_n because $x_{2^l;i} \equiv x_{0;i} + x_{0;2^l+i}$

3.5. All *sequences in the "null" tree on the* $2^l + 1$ *(*$l \in \mathrm{N}$*) floor is balanced.*

Sequence on 1 floor is equal e. Let sequence on 2^l+1 floor is $\alpha_1;\alpha_2;...;\alpha_n$. So we have that $\alpha_i+\alpha_{i+2^l}=1$. So one of these elements is 1 and one is zero. So it is simply to see that sequence is balanced. QED.

4. G-set method.

4.1. *A necessary and sufficient condition for 1-balanced sequence is the number of G-set is equal* $\frac{n}{2}$ *and sequence is 0-balanced sequence.*

Break the vector into G-sets. If we have k G-set after one operation we will have k element is equal 1. So if we want to have after one operation balanced sequence we must have n/2 G-set. In the other side it criterion is necessary and it is enough. If n don't divide by 4, we can see that we never don't have n/2 G-set, because G-set of element is equal 1 alternate with G-set of element is equal 0.So n divide by 4.

4.2. *A necessary and sufficient condition for 2-balanced sequence is the number of G-sets with a power of 1 is* $\frac{n}{4}$ *and condition for 1-balanced sequence.*

We represent our sequence as $(a_1)(a_2)(a_3)....(a_{n\backslash 2})$. If we rewrite our sequence $(a_1)(a_2)(a_3)....(a_{n\backslash 2})$ in its original form, it will look like this: first a_1 element is equal 1, then a_2 element is equal 0 and so on or first a_1 element is equal 0, then a_2 element is equal 1.This sequence's presentation is called "performance G-sets". Let sequence performances G-set. If all power of G-set will be integer then call this performance is "normal". We have that necessary and sufficient condition for 1-balanced sequence is the number of G-set is equal $\frac{n}{2}$. Let see how sequence will look after one operation: $(a_1-1)(1)(a_2-1)(1)...(a_{n\backslash 2}-1)(1)$. If we have G-set with power 0, then we *stick together* G-set with power 0 and its neighboring G-sets. If we have k G-set with a power 0, then the number of G-set in normal view reduced by two. If our G-set is 2-balanced then after one operation the number of G-set is equal $\frac{n}{2}$. So number of G-sets in this sequence with a power of 1 is $\frac{n}{4}$ QED.

4.3 *A necessary and sufficient condition for 3-balanced sequences is number of G-set with a power of 2 plus the number of neighboring G-set with a power (power of both G-set) greater than 1 is equal n/ 4 and condition for 2-balanced sequence.* (This obviously follows from previous statement).

5. Cycle

5.1. *Every power of G-set don't greater then power of cycle.*

5.2. *If power of cycle=k then* $C_{k-1}^{1}\alpha_{k;i+1}+\ldots+C_{k-1}^{k-1}\alpha_{k;i+k-1}\equiv 0$ *(mod 2)*

5.3 *If k=*2^{i} *(i be integer) then sequence is* e_n

5.4 *If we have one superbalanced sequence then we have infinitely superbalanced sequence.*

Let we have sequence A (it consists of n elements), so that A is superbalanced then we have superbalanced sequence consists of 2n element. We can *glue* 2 sequences A and new sequence will be superbalanced.

5.5 *If superbalanced have some equal under-sequence, then under-sequence is superbalanced.*

Let number of sequence A is minimal.

5.6 *If sequence A has condition that* $\sum_{i=j}^{l+j}\alpha_{0;i}\equiv 0$ *(for any j) , then our sequence A is not superbalanced.(l is integer and don't equal n)*

It simply to see that $\sum_{i=1}^{r}\alpha_{0;i}\equiv 0$ (element go one by one), $n\equiv r(\operatorname{mod} l)$ where (r don't greater then l), because sum of n element=0. Then analogically repeat this operation (we think that l:=r, and we find new r) and will repeat this when r will 1 or 0. If r=1 then sequence is equal "en". If r=0, then break our sequence for $\frac{n}{l}$ under-sequence by l element. It simply to see that all under-sequence will be equal. So we have sequence, so that number of element this sequence is smaller then number of elements sequence A. And this sequence is superbalanced.

Also we prove that power of cycle superbalanced sequence don't equal $2^{i}-2;2^{i}-1;2^{i}$

__Generalization__

We can generalization theorem 2.5($\alpha_{k;i}\equiv C_{k-1}^{0}\alpha_{0;i}+C_{k-1}^{1}\alpha_{k;i+1}+\ldots+C_{k-1}^{k-1}\alpha_{k;i+k-1}$ *(mod m)*).

__Theorem 5.1__

If sequence is 1-balanced then n divide by m^2.

If $a\equiv\beta_1+\beta_2(\operatorname{mod} m)$ then $a=\beta_1+\beta_2-mc$

It simply to see that only one product doesn't be equal 0. So every product will be equal 0 or x (where x is integer), then $k_{i;j}=\frac{k}{x}$ (where k=$\left[\frac{i+j}{m}\right]$). Let sequence $\alpha_{0;1};\alpha_{0;2};\ldots;\alpha_{0;n}$ is 1-balanced. And $a=\alpha_i+\alpha_j-c_{i;j}m$ if $a\equiv\alpha_1+\alpha_2(\operatorname{mod} m)$ (where $c_{i;j}$ is integer). We have that: $\sum_{i=1}^{n}\alpha_{0;i}=$ n/m $\sum_{i=1}^{n}\alpha_{1;i}=$ n/m. Then we have:

$$\sum_{i=1}^{n} \alpha_{1;i} = \sum_{i=1}^{n} \left(\alpha_{0;i} + \alpha_{0;i+1} - c_{i;i+1} m \right) = 2 \sum_{i=1}^{n} \alpha_{0;i} - -m \sum_{i=1}^{n} c_{i;i+1} =$$

$$= \frac{2n}{m} - m \sum_{i=1}^{n} c_{i;i+1} = \frac{n}{m} \Rightarrow m \sum_{i=1}^{n} c_{i;i+1} = \frac{n}{m}$$