4th International Tournament of Young Mathematicians

Belarus

Problem 6. Recurrent Sequences

Abstract

This problem is about studying such properties of some non-linear high-order recurrent sequences as monotonicity, bounds of a sequence, convergence or divergence. At the very beginning we were to study the sequence with every term being an arithmetic mean of squares of all previous terms (the first and the second terms were given). We've considered the more general case. In particular, we've examined the sequence with each term being a sum of α^{th} powers of previous terms, each summand multiplied be a term of some non-decreasing sequence of positive real numbers.

We've studied the sequence proposed in the second point of the problem and have proven its monotonicity for some values of u_1, u_2. We've presented an example of sequence (u_n) for which $(u_n)_{n\geq 20}$ and any $(u_n)_{n\geq t}$ $(t < 20)$ is not monotonic from the beginning. Also, we've found some explicit formulas for u_i for some values of the first two terms. We've also studied the case $u_1 < 0, u_2 > 0$.

We've proven that a sequence $u_{n+1} = \frac{u_1 u_{\sigma(1)} + \cdots + u_n u_{\sigma(n)}}{n}$ depending on u_1, u_2 and some permutation σ is convergent to 0 if $u_1^2 + u_2^2 < 2$ and diveregent if the product of two first terms is greater than 1.

Some results in studying the properties of sequences obtained by taking the arithmetic mean of summands of the form $u_i u_{f(i)}$ where $f: \mathbb{N} \to \mathbb{N}$ with some restrictions were obtained in the fifth part of the solution.

In the fourth part of the solution we studied the properties of sequence obtained by taking the arithmetic mean of summands of the form $u_i u_{i+1}$.

Finally, we've answered to the question of the second part of the fourth paragraph of the condition by proving that there exist infinitely many values of $u_4 \in \left[0; \frac{1}{3}\right]$, such that every term of the sequence starting from the forth one is less than 1. We've also studied some proprieties of this sequence.

We've also found the explicit formula for u_i for some values of u_1, u_2, u_3 and u_4.

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Sequence #1

Let $\alpha \in \mathbb{R}_+ \cup \{0\}$, $(m_n)_{n\geq 1}$ be a non-decreasing sequence (including a constant one) of positive numbers. Let $u_1, u_2 \in \mathbb{R}_+ \cup \{0\}$ (in order to avoid non-real values since α is arbitrary). Consider a sequence, satisfying the property that for all $n \geq 2$:

$$u_{n+1} = \frac{m_1 u_1^\alpha + m_2 u_2^\alpha + \cdots + m_n u_n^\alpha}{m_1 + m_2 + \cdots + m_n} \quad (1.1)$$

Then, for all $n \geq 2$:

$$u_{n+2} = \frac{(m_1 + \cdots + m_n)u_{n+1} + m_{n+1}u_{n+1}^\alpha}{m_1 + \cdots + m_{n+1}} = u_{n+1} + \frac{m_{n+1}(u_{n+1}^\alpha - u_{n+1})}{m_1 + \cdots + m_{n+1}} \quad (1.2)$$

<u>Theorem 1</u>:

1. $u_1 = u_2 = 0 \Rightarrow$ the sequence $(u_i)_{i\geq 3}$ is constant and its every term is equal to 0.
2. If $\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2} = 1 \vee \alpha = 0$ or $\vee\, \alpha = 1$ the sequence $(u_i)_{i\geq 3}$ is constant and its every term is equal to 1, 1, $\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2}$ respectively.
3. If $\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2} > 1$ and $\alpha > 1$, the sequence is strictly increasing from the third term (and bounded below by the third term) and divergent.
4. If $\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2} < 1$ and $\alpha > 1$, the sequence is strictly decreasing from the third term (and bounded above by the third term) and convergent to zero and bounded below by it.
5. If $\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2} > 1$ and $\alpha < 1$, the sequence is strictly decreasing from the third term (and bounded above by the third term), bounded below by 1 and convergent.
6. If $\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2} < 1$ and $\alpha < 1$, the sequence is strictly increasing from the third term (and bounded below by the third term), bounded above by 1 and convergent.

Proof:

<u>Remark 1</u>. "The sequence is strictly decreasing/increasing" refers to the $(u_n)_{n\geq 3}$

$$\frac{m_1u_1^\alpha + m_2u_2^\alpha}{2} = u_3$$

1) Suppose both u_1, u_2 are zeroes. In this case, $\forall u_i = 0$. Since now not both u_1, u_2 are zeroes.
2) $u_3 = 1$. In this case $u_4 = 1$, and for every $u_l = 1$ using (1.2) $u_{l+1} = u_l = 1$.
   $\alpha = 1 \Rightarrow (1.2) \Rightarrow u_n = u_{n-1} = \cdots = u_3 \; \forall n > 3$
   $\alpha = 0 \Rightarrow (1.1) \Rightarrow u_i = 1$
3) $u_3 > 1, \alpha > 1$. For every $u_l > 1$ using (1.2) $u_{l+1} > u_l > 1$. I.e. the sequence is strictly increasing and bounded below by u_3.
   Let $u_3^{\alpha-1} = 1 + \varepsilon \Rightarrow u_i^{\alpha-1} - 1 > \varepsilon \; \forall i > 3$. Then (for sufficiently large k):
   $u_{k+1} = u_k + \frac{m_k u_k(u_k^{\alpha-1}-1)}{m_1+\cdots+m_k} \geq u_k + \frac{u_k(u_k^{\alpha-1}-1)}{k} > u_k + \frac{u_k\varepsilon}{k} > u_k + \frac{\varepsilon}{k} > u_{k-1} + \varepsilon\left(\frac{1}{k} + \frac{1}{k-1}\right) > u_3 + \varepsilon\left(\frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{k}\right) > \varepsilon\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{k}\right) - \varepsilon\left(\frac{1}{2} + \frac{1}{3}\right) = \varepsilon \cdot \sum_1^k \frac{1}{i} - c.$

$$\lim_{n\to\infty} u_{n+1} > \lim_{n\to\infty}\left(\varepsilon\cdot\sum_{1}^{n}\frac{1}{i}-c\right)=\varepsilon\cdot\lim_{n\to\infty}\left(\sum_{1}^{n}\frac{1}{i}\right)-c=\infty \quad [1]$$

4) $u_3 < 1, \alpha > 1 \Rightarrow (1.2) \Rightarrow (u_n)$ is strictly decreasing. It's obviously bounded below by 0 and bounded above by u_3. Consequently, it's convergent.

Let $u_3^{\alpha-1} = 1-\varepsilon, \varepsilon \in (0;1)$. Then $1-u_i^{\alpha-1} > \varepsilon\ \forall i > 3$ and for sufficiently large k:

$$u_{k+1} = u_k\left(1-\frac{1-u_k^{\alpha-1}}{m_1+\cdots+m_k}\cdot m_k\right) \le u_k\left(1-\frac{1-u_k^{\alpha-1}}{k}\right) < u_k\left(1-\frac{\varepsilon}{k}\right)$$

$$< u_{k-1}\left(1-\frac{\varepsilon}{k-1}\right)\left(1-\frac{\varepsilon}{k}\right) < \cdots < u_3\prod_{i=3}^{k}\left(1-\frac{\varepsilon}{i}\right) = \frac{\prod_{i=1}^{k}\left(1-\frac{\varepsilon}{i}\right)}{1-\frac{\varepsilon}{2}}$$

$$= C\cdot\prod_{i=1}^{k}\left(1-\frac{\varepsilon}{i}\right)$$

Let $a_k = \frac{\varepsilon}{k}$. Obviously, $0 < a_k < 1$. The a_k series is divergent because the harmonic series is divergent. In this case, the infinite product $\prod_{i=1}^{\infty}(1-a_i) = 0$. Proof:

$$\prod_{i=1}^{\infty}(1-a_i) = 0 \Leftrightarrow \text{[taking the logarithm]} \Leftrightarrow \sum_{i=1}^{\infty}\ln(1-a_i) = -\infty$$

I.e. it's sufficient to prove that $-\ln(1-a_k)$ series is divergent. Using the Limit Comparison Test and the fact that a_k divergents:

$$\lim_{k\to\infty}\frac{-\ln(1-a_k)}{a_k} = \lim_{x\to -0}\frac{\ln(1+x)}{x} = 1$$

Since the harmonic series is divergent, $-\ln(1-a_k)$ series is also divergent $\Rightarrow \sum_{i=1}^{\infty}\ln(1-a_k) = -\infty \Rightarrow \prod_{i=1}^{\infty}(1-a_i) = 0$. But $0 < u_k < C\cdot\prod_{i=1}^{k-1}(1-a_i)$ for sufficiently large k, and according to the sandwich rule $\lim_{n\to\infty} u_n = 0$ ■

Corollary 1. If a sequence depending on u_1, u_2 satisfies $u_{n+1} = \frac{u_1^\alpha u_{\sigma(1)}^\beta+\cdots+u_n^\alpha u_{\sigma(n)}^\beta}{n}$ $(\alpha,\beta > 0, \alpha+\beta > 1)$ where σ is some permutation and $u_1^{\alpha+\beta} + u_2^{\alpha+\beta} < 2$, then it's convergent to zero. This result can be easily obtained using the Rearrangement inequality.

5) $\alpha < 1, u_3 > 1$. Suppose $u_k > 1$. Then $u_{k+1} = \frac{(m_1+\cdots+m_{k-1})u_k+m_k u_k^\alpha}{m_1+\cdots+m_k} > \frac{(m_1+\cdots+m_{k-1})+m_k}{m_1+\cdots+m_k} = 1$. Also: $u_{k+1} = \frac{(m_1+\cdots+m_{k-1})u_k+m_k u_k^\alpha}{m_1+\cdots+m_k} < \frac{(m_1+\cdots+m_{k-1})+m_k}{m_1+\cdots+m_k}u_k = u_k \Rightarrow$ The sequence is strictly decreasing, bounded below by 1 and bounded above by u_3. Consequently, it's convergent.

6) $\alpha < 1, u_3 < 1. \Rightarrow$ If $u_k < 1$: $u_{k+1} = \frac{(m_1+\cdots+m_{k-1})u_k+m_k u_k^\alpha}{m_1+\cdots+m_k} < \frac{(m_1+\cdots+m_{k-1})+m_k}{m_1+\cdots+m_k} < 1$. Also: $u_{k+1} = \frac{(m_1+\cdots+m_{k-1})u_k+m_k u_k^\alpha}{m_1+\cdots+m_k} > \frac{(m_1+\cdots+m_{k-1})+m_k}{m_1+\cdots+m_k}u_k = u_k \Rightarrow$ The sequence is strictly increasing, bounded above by 1 and bounded below by u_3. Consequently, it's convergent.

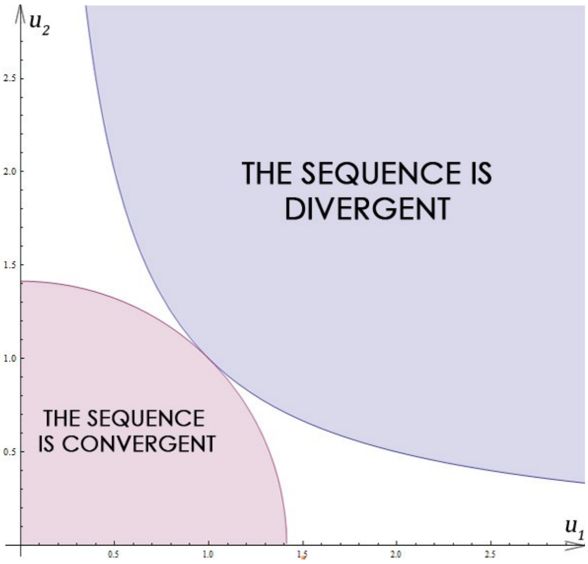
Corollary 2. The case $(m_i) = 1, \alpha = 2$ is a partial case of Theorem 1.

Sequence #2

Suppose for all $n \geq 2$:

$$u_{n+1} = \frac{u_1^{\mu} u_{\sigma(1)}^{2-\mu} + u_2^{\mu} u_{\sigma(2)}^{2-\mu} + \cdots + u_n^{\mu} u_{\sigma(n)}^{2-\mu}}{n}$$

Where σ is a random permutation, $\mu \in \mathbb{R}$ and $u_1, u_2 > 0$ are given.



Theorem 2. Not depending on the permutation, the sequence with:

1. $u_1 = u_2 = 0$ or 1 is constant and its every term is equal to 0 or 1 respectively;
2. $u_1^2 + u_2^2 < 2$ is convergent to zero;
3. $u_1 \cdot u_2 > 1$ is divergent.

Proof:

1) Consider the trivial case $u_1 = u_2 = 0$, $u_1 = u_2 = 1$. Then every term is equal to 0 or 1 respectively.
2) Corollary 1 $\Rightarrow$ if $u_1^2 + u_2^2 < 2$ $\lim_{n\to\infty} u_n = 0$. Note that if $u_k = u + \varepsilon$ $(u, \varepsilon > 0)$, since every $u_i > 0$, setting $u_k = u$ will only reduce every following term.
3) Let $u_1 = \frac{\alpha}{k}, u_2 = k\alpha^2$ $(\alpha > 1, k > 0)$. Then:
   $u_3 \geq \frac{2\sqrt{u_1^{\mu} u_2^{\mu} u_{\sigma(1)}^{2-\mu} u_{\sigma(2)}^{2-\mu}}}{2} =$ [since σ is a permutation] $= \sqrt{\alpha^{3\cdot 2}} = \alpha^3$. Similarly, if $\forall\ 3 \leq l_0 \leq l$ $u_{l_0} \geq \alpha^{l_0}, u_{l+1} \geq \frac{l\sqrt[l]{(u_1 \ldots u_l)^{\mu} \cdot (u_{\sigma(1)} \ldots u_{\sigma(l)})^{2-\mu}}}{l} \geq \sqrt[l]{\alpha^{\frac{l(l+1)}{2}\cdot 2}} = \alpha^{l+1}$. Since $\alpha > 1$, $\lim_{n\to\infty} u_n \geq \lim_{n\to\infty} \alpha^n = \infty$.
   So, for all fixed $\alpha > 1\ \forall\ k > 0$ the sequence with $u_1 \geq \frac{\alpha}{k}, u_2 \geq k\alpha^2$ is divergent. Let $\alpha \to 1$. Then we obtain the desired result, i.e. if $u_1 \cdot u_2 > 1$, the sequence is divergent.

Sequence #3

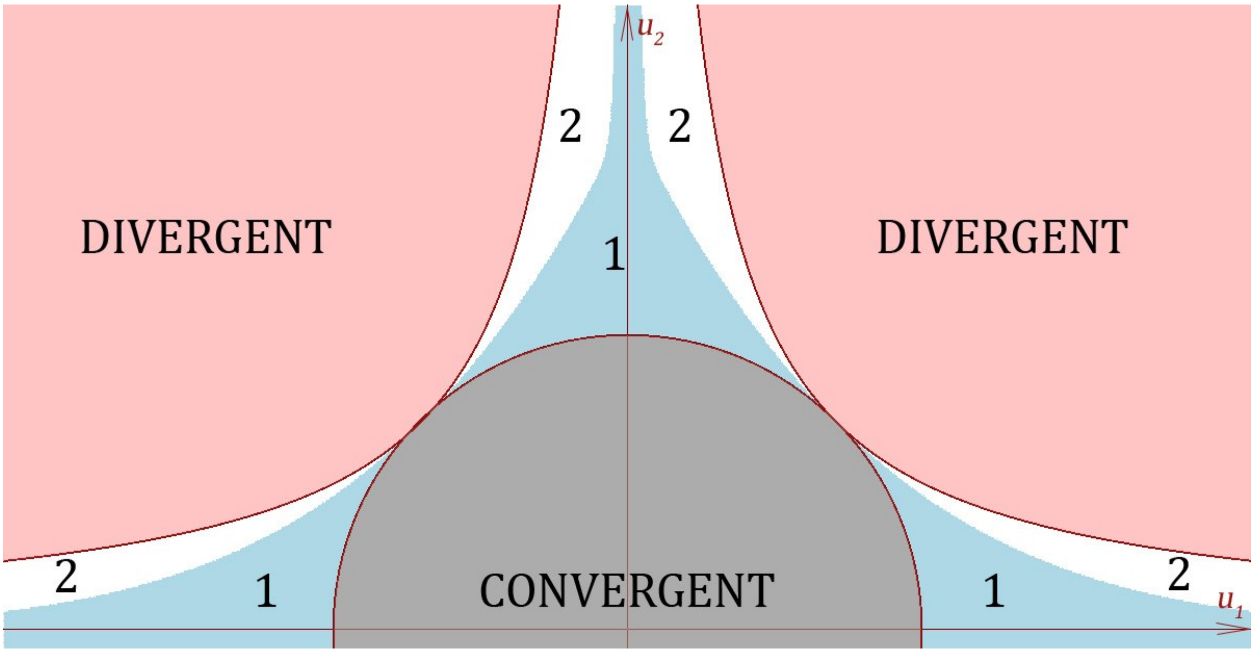
Consider a sequence

$$u_{n+1} = \frac{u_1 u_n + u_2 u_{n-1} \ldots + u_n u_1}{n}$$

Theorem 3. This sequence is:

1) Constant if $u_1 u_2 = 0$ or $u_1 = u_2 = 1$ (every $u_{i\geq 3} = 0$ or 1 respectively)

2) Strictly increasing from the third term if $u_1, u_2 \geq 1$ ($u_1 u_2 \neq 1$) (it's bounded below by $u_3 > 1$ and divergent (#2))
3) Strictly decreasing from the third term if $u_1, u_2 \leq 1$ ($u_1 u_2 \neq 1$) (it's bounded above by $u_3 < 1$, bounded below by 0 and convergent to 0 (Corollary 1).
4) We've also proven that if $x_1, x_2 > 0$ and $u_1 = -x_1, u_2 = x_2,\ u_{n+1} = \frac{1}{n}\sum_{i=1}^{n} u_i u_{n+1-i}$, $x_{n+1} = \frac{1}{n}\sum_{i=1}^{n} x_i x_{n+1-i}$, then $x_i = (-1)^i u_i$.
5) We've also constructed an example of sequence for which there is no monotonicity among the first ~20 terms.
6) We've also found the explicit formula for u_i ($u_i = \alpha^i$) if $\exists \alpha \in \mathbb{R}: u_1 = \alpha \,\&\, u_2 = \alpha^2$.



Zone 1 – experiments show that there the sequence is convergent

Zone 2 – experiments show that the sequence is divergent

<u>Proof:</u>

1) Suppose $u_1 = u_2 = 1 \vee u_1 u_2 = 0$. Obviously, in this case every term of the sequence is equal to 1 or 0 respectively.
2) Suppose $1 \leq u_1 \leq u_2 \,\&\, u_2 \neq 1$. Then $u_3 = u_1 u_2 \geq u_2, u_4 = u_2\left(\frac{2}{3}u_1^2 + \frac{u_2}{3}\right) > u_1 u_2 = u_3$. Suppose for some k $1 \leq u_1 \leq u_2 \leq u_3 < \cdots < u_{(k+1)\geq 4}$ (2.1). Then:

$$u_{k+2} > u_{k+1} \Leftrightarrow k u_1 u_{k+1} + k u_2 u_k + \cdots + k u_{k+1} u_1 > (k+1) u_1 u_k + \cdots + (k+1) u_k u_1 \quad (2.2)$$

$$(2.1) \Rightarrow k u_1 u_{k+1} + u_2 u_k > (k+1) u_1 u_k$$

$$(k-1) u_2 u_k + 2 u_3 u_{k-1} > (k+1) u_2 u_{k-1}$$

$$\ldots$$

$$(k+1-i) u_i u_{k+2-i} + i u_{i+1} u_{k+1-i} > (k+1) u_i u_{k+1-i}$$

$$u_k u_2 + k u_{k+1} u_1 > (k+1) u_k u_1 \ \blacksquare$$

In this case the sequence $(u_n)_{n\geq 3}$ is strictly increasing, bounded below by u_3 and divergent (#2).

Suppose $1 \leq u_2 < u_1$ & $u_1 \neq 1$. Then $u_3 = u_1 u_2 \geq u_2, u_4 = u_2\left(\frac{2}{3}u_1^2 + \frac{u_2}{3}\right) > u_1 u_2 = u_3$.

$$2u_1^2 - 3u_1 + u_2 \geq 2u_1^2 - 3u_1 + 1 > 0\ \forall u_1 > 1.$$

Suppose for some k $1 \leq u_2 \leq u_1 \leq u_3 < \cdots < u_{(k+1)\geq 4}$ (2.3). Then:

$$u_{k+2} > u_{k+1} \Leftrightarrow ku_1u_{k+1} + ku_2u_k + \cdots + ku_{k+1}u_1 > (k+1)u_1u_k + \cdots + (k+1)u_ku_1 \text{ (2.2)}$$

$$(2.1) \Rightarrow ku_2u_k + u_1u_{k+1} > (k+1)u_1u_k$$

$$(k-1)u_1u_{k+1} + 2u_3u_{k-1} > (k+1)u_2u_{k-1}$$

$$(k-2)u_3u_{k-1} + 3u_4u_{k-2} > (k+1)u_3u_{k-2}$$

$$(k+1-i)u_iu_{k+2-i} + iu_{i+1}u_{k+1-i} > (k+1)u_iu_{k+1-i}$$

$$3u_{k-2}u_4 + (k-2)u_{k-1}u_3 > (k+1)u_{k-2}u_3$$

$$2u_{k-1}u_3 + (k-1)u_{k+1}u_1 > (k+1)u_{k-1}u_2$$

$$u_{k+1}u_1 + ku_ku_2 > (k+1)u_ku_1 \ \blacksquare$$

In this case the sequence $(u_n)_{n\geq 3}$ is strictly increasing and bounded below by u_3.

Using the results obtained in (#2), the sequence is divergent if the product of two first terms is greater than one.

3) Suppose $0 < u_1, u_2 < 1$. Similarly to the proof above one can easily prove that (u_n) is strictly decreasing. (#1) $\Rightarrow u_n$ is convergent to zero.
4) Suppose $u_1 < 0, u_2 > 0$. Then, obviously: $u_3 < 0$ and each u_i has the same sign as $(-1)^i$. Really, suppose for some m_0 our statement holds. Then $u_{m_0+1} = \frac{1}{m_0}\sum_{i=1}^{m_0} u_i u_{m_0+1-i}$. So, if $m_0 \vdots 2$, then u_i and u_{m_0+1-i} have different parity and each summand is negative. Else each summand is positive. Moreover, if we construct a sequence x_n in the same way depending $x_1 = |u_1|, x_2 = u_2, \forall i\colon |u_i| = x_i$, i.e. if x_i is divergent (for example, as proven in (#2), $x_1x_2 > 1$), so is u_1. And even $u_i = (-1)^i x_i$.

Corollary 2. If $u_1 = \alpha \in \mathbb{R}, u_2 = \alpha^2$, then $u_i = \alpha^i$. This result can also be simply proven using induction.

5) Consider a sequence with $u_1 = 0.1, u_2 = 100$. We've calculated some of the first terms:

0.1, 100, 10, 3334, 666.7, 133407, 37783.6, $5.40254 \cdot 10^{006}$, $1.96886 \cdot 10^{006}$, $2.19073 \cdot 10^{008}$, $9.75463 \cdot 10^{007}$, $8.88594 \cdot 10^{009}$, $4.67343 \cdot 10^{009}$, $3.6049 \cdot 10^{011}$, $2.18616 \cdot 10^{011}$, $1.4627 \cdot 10^{013}$, $1.00453 \cdot 10^{013}$, $5.93588 \cdot 10^{014}$, $4.55211 \cdot 10^{014}$, $2.40927 \cdot 10^{016}$, $2.04008 \cdot 10^{016}$, $9.78035 \cdot 10^{017}$, $9.06041 \cdot 10^{017}$
It's easy to see that there's no monotonicity among the first ~20 terms.

Sequence #4

Consider a sequence

$$u_{n+1} = \frac{u_1 u_2 + u_2 u_3 \dots + u_n u_1}{n} \quad (2.1)$$

For all $n \geq 2$ and u_1, u_2 given. In this case:

$$u_{n+2} = \frac{u_1 u_2 + u_2 u_3 + \cdots + u_n u_{n+1} + u_{n+1} u_1}{n+1} = \frac{n u_{n+1} - u_1 u_n + u_n u_{n+1} + u_1 u_{n+1}}{n+1} =$$
$$= \frac{n + u_n}{n+1} u_{n+1} + \frac{u_1 (u_{n+1} - u_n)}{n+1} \quad (2.2)$$

If $u_1 u_2 = 0 \ \forall u_{i \geq 3} = 0$. If $u_1 = u_2 = 1 \ \forall u_{i \geq 3} = 1$.

Suppose $u_1 > 0$ and at some point $1 < u_n \leq u_{n+1}$ or $1 \leq u_n < u_{n+1}$. Then (2.2) $\Rightarrow$ either $u_{n+2} \geq \frac{n+u_n}{n+1} u_{n+1} > u_{n+1}$ or $u_{n+2} > \frac{n+u_n}{n+1} u_{n+1} \geq u_{n+1}$. The result is the same: $u_{n+2} > u_{n+1}$ (2.3).

So, if $u_1, u_2 \geq 1$ & $u_1 u_2 > 1$: $u_3 = u_1 u_2$ & either $u_3 \geq u_2 > 1$ or $u_3 > u_2 \geq 1$. In this case (2.3) $\Rightarrow (u_n)_{n \geq 3}$ is strictly increasing.

If $u_1, u_2 \geq 1$ & $u_1 u_2 > 1$ the sequence (2.1) (for $n \geq 3$) is strictly increasing and bounded below by u_3. (#2) $\Rightarrow$ $u_1 u_2 > 1 \Rightarrow$ the sequence is divergent.

Suppose $0 < u_1, u_2 \leq 1$ & $u_1 u_2 < 1$. Then, similarly to (2.3), we obtain $u_{n+2} < u_{n+1}$ (2.4). Using Corollary 1, this sequence, like any other sequence of this form with $u_1^2 + u_2^2 < 2$, is convergent to zero.

Sequence #5

Let $f: \mathbb{N} \to \mathbb{N}$ satisfy the following property: $f(n) \leq n \ \forall \ n \geq 2$ and $f(1) \leq 2$ (5.0).

Consider a sequence depending on u_1, u_2:

$$u_{n+1} = \frac{u_1 u_{f(1)} + u_2 u_{f(2)} \dots + u_n u_{f(n)}}{n} \quad (5.0)$$

For all $n \geq 2$. In this case we say that f *constructs* the sequence.

<u>Theorem 5.</u>

1) $u_1 = u_2 = 1 \Rightarrow u_i = 1, u_1 = u_2 = 0 \Rightarrow u_i = 0$
2) If $u_1, u_2 > 1$, the sequence is strictly increasing, bounded below by u_3 and divergent not depending on f.
3) If $0 < u_1, u_2 < 1$, the sequence is strictly decreasing, bounded below by 0, bounded above by u_3 and convergent to 0 not depending on f.
4) If one of u_1, u_2 equals 1, if the other is greater than 1, the sequence is non-decreasing, else (i.e. the other term is less than zero but greater than one) it's non-increasing.

5) If one of the first terms is greater than 1 , and the other belongs to $[0; 1)$, then the sequence can be both decreasing and convergent to 0 and increasing and divergent.
6) If some terms are negative, the sequence can both converge to 0 and diverge to $\pm\infty$ depending on u_1, u_2.

Some other cases were investigated and some probabilities were calculated.

Proof:

Taking into account that $u_{n+1} = \frac{u_1 u_{f(1)} + u_2 u_{f(2)} \dots + u_n u_{f(n)}}{n}$ $(n \geq 2)$ we can rewrite (5.0) as follows:

$$u_{n+2} = \frac{n u_{n+1} + u_{n+1} u_{f(n+1)}}{n+1} = \frac{u_{n+1}\left(n + u_{f(n+1)}\right)}{n+1} \quad (5.1)$$

It's easy to see that $u_{f(n)} > 1 \Rightarrow u_{n+1} > u_n$

Consider a few cases:

1) Obviously, if $u_1 = u_2 = 1 \Rightarrow u_i = 1, u_1 = u_2 = 0 \Rightarrow u_i = 0$
2) $u_1, u_2 > 1 \Rightarrow \forall u_i > 1 \Rightarrow (5.1) \Rightarrow (u_n) \nearrow$. The sequence is strictly increasing and it's also easy-to-verify that it's divergent: obviously $u_{i \geq 3} > \max\{u_1, u_2\} \geq \min\{u_1, u_2\} = 1 + \alpha$

$$u_{k+2} \geq \frac{u_{k+1}(k+1+\alpha)}{k+1} = u_{k+1}\left(1 + \frac{\alpha}{k+1}\right) > u_k\left(1 + \frac{\alpha}{k+1}\right)\left(1 + \frac{\alpha}{k}\right) > \cdots$$
$$> u_3 \prod_{i=3}^{k+1}\left(1 + \frac{\alpha}{i}\right)$$

Now we'll show that $\lim_{n\to\infty} \prod_1^{n+1}\left(1 + \frac{\varepsilon}{i}\right) = \infty$

$$\prod_{i=1}^{\infty}\left(1 + \frac{\varepsilon}{i}\right) = \infty \Leftrightarrow \ln \prod_{i=1}^{\infty}\left(1 + \frac{\varepsilon}{i}\right) = \infty \Leftrightarrow \sum_{i=1}^{\infty} \ln\left(1 + \frac{\varepsilon}{i}\right) = \infty$$

Similarly to (#1.4) using Limit Comparison Test and the fact that the harmonic series is divergent:

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{\varepsilon}{n}\right)}{\frac{\varepsilon}{n}} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 \Rightarrow \sum_{i=1}^{\infty} \ln\left(1 + \frac{\varepsilon}{i}\right) = \infty \Rightarrow \prod_{i=1}^{\infty}\left(1 + \frac{\varepsilon}{i}\right) = \infty \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} u_n > c \cdot \prod_{i=1}^{\infty}\left(1 + \frac{\varepsilon}{i}\right) = \infty \Rightarrow \lim_{n\to\infty} u_n = \infty$$

3) $0 < u_1, u_2 < 1 \Rightarrow \forall u_i < 1 \Rightarrow (5.1) \Rightarrow (u_n) \searrow$. The sequence is strictly decreasing (not depending on f) and it's also easy-to-verify that it's convergent: obviously $u_{i \geq 3} < \max\{u_1, u_2\} = 1 - \alpha$

$$u_{k+1} \leq \frac{u_k(k-\alpha)}{k} = u_k\left(1 - \frac{\alpha}{k}\right) \leq u_{k-1}\left(1 - \frac{\alpha}{k-1}\right)\left(1 - \frac{\alpha}{k}\right) \leq \cdots \leq u_3 \prod_{i=3}^{k}\left(1 - \frac{\alpha}{i}\right)$$

The fact that $\prod_{i=1}^{\infty}\left(1 - \frac{\varepsilon}{i}\right) = 0$ $(\varepsilon \in (0; 1))$ has been already proven in (#1) $\Rightarrow \lim_{n\to\infty} u_n = 0$.

4) $u_1 \vee u_2 = 1$ & $u_1 \cdot u_2 \neq 1$. Since only one of term is equal to 1, the probability of $u_{n+1} = u_n$ is equal to $\frac{1}{n}$. If $u_1 \cdot u_2 > 1$ the sequence is non-decreasing, and if $0 < u_1 \cdot u_2 < 1$ the sequence in non-increasing. The probability of $u_{k+\alpha} = u_{k\geq 3}$ is equal to $\frac{1}{k\cdot(k+1)\cdot\ldots\cdot(k+\alpha-1)}$.

5) One of $u_1, u_2 \in \mathbb{R}_+ \cup \{0\}$ is greater than one and another is less than one. Consider the function $g(n)$ constructed as follows:

$$g(n) = \begin{cases} 1, & u_n \geq 1 \\ 0, & u_n < 1 \end{cases}$$

The probability of $u_{n+1} \geq u_n$ is equal to $\frac{g(n)+g(n-1)+\cdots+g(3)+1}{n} \leq 1 - \frac{1}{n}$.

Let us prove that in this case the sequence can be convergent. WLOG, suppose that $0 \leq u_1 = \zeta < 1, u_2 > 1$. Consider a function $f(n) = 1$. Obviously, it satisfies (5.0). Then for sufficiently large n:

$$u_{n+2} = \frac{u_{n+1}(n+\zeta)}{n+1} = u_{n+1}\left(1 - \frac{1-\zeta}{n+1}\right) = u_n\left(1 - \frac{1-\zeta}{n}\right)\left(1 - \frac{1-\zeta}{n+1}\right) = \cdots$$
$$= u_3 \prod_{i=3}^{n+1}\left(1 - \frac{1-\zeta}{i}\right) = c \cdot \prod_{i=1}^{n+1}\left(1 - \frac{1-\zeta}{i}\right)$$

The fact that $\prod_{i=1}^{\infty}\left(1 - \frac{1-\zeta}{i}\right) = 0$ $(1-\zeta \in (0;1))$ has been already proven in (#1). If $1 - \zeta = 1$, then $\prod_{i=3}^{n+1}\left(1 - \frac{1-\zeta}{i}\right) = \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n}{n+1} \cdot \ldots = \lim_{n\to\infty} \frac{2}{n} = 0$. So, $\lim_{n\to\infty} u_n = 0$.

Also, $u_{n+2} = \frac{u_{n+1}(n+\zeta)}{n+1} < u_{n+1}$, so the sequence is decreasing.

Let us prove that in this case the sequence can be divergent. WLOG, suppose that $0 \leq u_1 < 1, u_2 = \mu + 1 > 1$. Consider a function $f(n) = 2$. Obviously, it satisfies (5.0). Then for sufficiently large n:

$$u_{n+2} = \frac{u_{n+1}(n+1+\mu)}{n+1} = u_{n+1}\left(1 + \frac{\mu}{n+1}\right) = u_n\left(1 + \frac{\mu}{n}\right)\left(1 + \frac{\mu}{n+1}\right) = \cdots$$
$$= u_3 \prod_{i=3}^{n+1}\left(1 + \frac{\mu}{i}\right) = c \cdot \prod_{i=1}^{n+1}\left(1 + \frac{\mu}{i}\right)$$

The fact that $\prod_{i=1}^{\infty}\left(1 + \frac{\mu}{i}\right) = \infty$ has been already proven in (#4.2). Then $\Rightarrow \lim_{n\to\infty} u_n = \infty$.

Also, $u_{n+2} = \frac{u_{n+1}(n+1+\mu)}{n+1} > u_{n+1}$, so the sequence is increasing.

6) $u_1 < 0 \vee u_2 < 0 \vee u_1, u_2 < 0$. If f is a function that constructs this sequence and $x_1 = |u_1|, x_2 = |u_2|$, consider a sequence (x_n) constructed as follows: $x_{n+2} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i x_{f(i)}$. Taking into account one of the properties of modules we easily conclude: $|u_i| < |x_i|$. This means that if (x_i) is convergent, so is (u_i). For example, if $-1 < u_1, u_2 < 0$, then the sequence is convergent to zero.

The sequence can also diverge to $\pm\infty$. For example, take $u_1^2 + u_2^2 > 2$ and $f(n) = n$. Then, according to (#1), $\lim_{n\to\infty} u_n = +\infty$ and the sequence is strictly increasing from the third term.

Let $u_1 > 1, u_2 < -1, f(1) = 2, f(n) = 1$ $\forall n \geq 2$. Then $u_3 = u_1 u_2 < -1$. Since $f(n) = 1, n \geq 2$ and (5.1), $u_4 < u_3$. Using simple induction, one can easily prove that $u_n < u_{n-1} < \cdots < u_3 < 0$. Similarly to (#5.2), the sequence is divergent to $-\infty$.

Sequence #6

a.

$$u_n = \frac{u_1u_{n+1} + u_2u_n + \cdots + u_{n+1}u_1}{n+1} \quad (5.1)$$

Let $u_1 = \frac{1}{\alpha}, u_2 = 1, u_3 = \alpha, u_4 = \alpha^2$. Then, $u_4 = \alpha^2 = \frac{1}{5}\left(\frac{2u_5}{\alpha} + 3\alpha^2\right) \Rightarrow u_5 = \alpha^3$. Suppose for all $i \leq m$: $u_i = \alpha^{i-2}$. Then $u_m = \alpha^{m-2} = \frac{1}{m+1}\left(\frac{2u_{m+1}}{\alpha} + (m-1)\alpha^{m-2}\right) \Rightarrow u_{m+1} = \alpha^{m-1}$.

b.

Notice #1. $\nexists\, u_4 \in \mathbb{R}: u_n < 1\ \forall\, n \geq 1$ because $u_1 = u_2 = u_3 = 1$ according to the condition. Therefore, we'll rephrase the condition: $\exists?\ 0 \leq u_4 < 1: u_n < 1\ \forall\, n \geq 4$?

Theorem 6.

1. We'll obtain the explicit formula for u_{n+1} $(n \geq 4)$.

$$u_n = \frac{u_1u_{n+1} + u_2u_n + \cdots + u_{n+1}u_1}{n+1} \overset{u_1=u_2=u_3=1}{\Longleftrightarrow} 2u_{n+1}$$
$$= (n-1)u_n - 2u_{n-1} - \sum_{4}^{n-2} u_iu_{n+2-i} \quad (4.1)$$

Set $u_4 = \alpha$:

$$u_{n+1} = \frac{1}{2}\left((n-1)u_n - 2u_{n-1} - 2\alpha u_{n-2} - \sum_{5}^{n-3} u_iu_{n+2-i}\right) \quad (4.2)$$

2. We'll calculate u_5, u_6, u_7.

$$u_4 = \alpha = \frac{2u_1u_5 + 2u_2u_4 + u_3^2}{5} \Leftrightarrow 2u_5 + 2\alpha + 1 = 5\alpha \Rightarrow u_5 = \frac{3\alpha - 1}{2}$$
$$u_5 = \frac{3\alpha - 1}{2} = \frac{2u_1u_6 + 2u_2u_5 + 2u_3u_4}{6} \Leftrightarrow \frac{2u_6 + 5\alpha - 1}{6} \Rightarrow u_6 = 2\alpha - 1$$
$$u_6 = 2\alpha - 1 = \frac{2u_1u_7 + 2u_2u_6 + 2u_3u_5 + u_4^2}{7} \Leftrightarrow 14\alpha - 7 = 2u_7 + 4\alpha - 2 + 3\alpha - 1 + \alpha^2 \Rightarrow$$
$$\Rightarrow u_7 = \frac{1}{2}(7\alpha - \alpha^2 - 4) \leq \frac{1}{2}(7\alpha - 4)$$

3. We'll prove that $u_n < 1\ \forall n \geq 4$

   Let $\alpha \in \left[0; \frac{1}{3}\right]$. Then $0 \geq u_5 > u_6 > u_7$. Suppose $0 \geq u_5 > u_6 > \cdots > u_k$ for some $k > 6$. We'll prove that $u_k > u_{k+1} \Leftrightarrow 0 > 2(u_{k+1} - u_k)$. $2(u_{k+1} - u_k) = (k-3)u_k - 2u_{k-1} - 2\alpha u_{n-2} - (u_5u_{k-3} + \cdots + u_{k-3}u_5) > (k-6)u_k - (u_5u_{k-3} + \cdots + u_{k-3}u_5)$. It's easy to see that $(k-6)u_k < 0$ and $(u_5u_{k-3} + \cdots + u_{k-3}u_5) > 0$ since every summand is positive. ■

   So, for every $u_4 \leq \frac{1}{3}$ the sequence $(u_n)_{n\geq 5}$ will be strictly decreasing, bounded above by $u_5 \leq 0$. Since $0 \geq u_5 > u_6 > \cdots$ (and $|u_i| < |u_{i+1}|\ \forall\, i \geq 5$):

$$|u_{k+1} - u_k| = u_k - u_{k+1} = -\left(\frac{1}{2}\big((k-3)u_k - 2u_{k-1} - 2\alpha u_{k-2} - (u_5 u_{k-3} + \cdots + u_{k-3}u_5)\big)\right)$$
$$> -\frac{1}{2}(k-6)u_k > \frac{1}{2}|u_k| > \frac{1}{2}|u_6| = \varepsilon > 0 \Rightarrow u_{k+l} < u_k - l \cdot \varepsilon \Rightarrow \lim_{n\to\infty} u_n$$
$$= -\infty.$$

■