4th International Tournament of Young Mathematicians

Belarus

Problem 5. Stable Polygons

Abstract

This problem is about calculating the cardinality of subsets of roots of unity with vanishing sum for fixed n. Moreover, one of the tasks of this problem was to describe all sets with vanishing sums.

This problem is divided into some subcases depending on the factorization of n.

In our solution we used some properties of roots of unity. Also, we've based some part of our solution on two works: On Vanishing Sums of Roots of Unity by T. Y. Lam, K. H. Leung and On Vanishing Sums of Distinct Roots of Unity by Gary Sivek.

Using this articles, we can easily obtain all values of k for which there exists a vanishing sum of k n^{th} roots of unity depending on the factorization of n.

Let $n \geq 3$ be a positive integer, and let P_n be the set of vertices of a regular $n-$gon. A subset $A \subset P_n$ is called stable if the center of gravity of the points in A coincides with the center ofthe regular $n-$gon.

Figure 2. The subset $A = \{A_1, A_2, A_3\}$ of P_{12} is stable, but the subset $B = \{B_1, ... , B_6\}$ is not.

1. When n is prime, find the number of stable subsets A _ P_n and describe them.

2. The same problem when n is the product of two distinct prime numbers.

3. The same problem when n is a power of a prime number.

4. Investigate the problem for an arbitrary n.

5. Suggest and study additional directions of research.

Fact 0. The condition of the problem is equivalent to the following: describe all sets of n^{th} roots of unity with vanishing sums. This fact follows from the basic properties of complex numbers.

Fact 1. If the repetition of roots is allowed and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ then n is k-balancing if and only if $k \in \mathbb{N}_0 p_1 + \mathbb{N}_0 p_2 + \cdots + \mathbb{N}_0 p_l$. This fact has been proven in [1].

Fact 2. If the repetition of roots is not allowed and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ then n is k-balancing if and only if $k \in \mathbb{N}_0 p_1 + \mathbb{N}_0 p_2 + \cdots + \mathbb{N}_0 p_l$ and $n - k \in \mathbb{N}_0 p_1 + \mathbb{N}_0 p_2 + \cdots + \mathbb{N}_0 p_l$. This fact has been proven in [2].

$\zeta = e^{\frac{2\pi i}{n}}$

Case 1. $n = p$

Then the only balancing subset is the whole set of n^{th} roots of unity and the empty set if we assume that the center of gravity of an empty set coincides with 0. The proof immediately follows from [2].

Case 2. $n = pq$

In [2] it has been proven that in this case if one can find a k-balancing set then $k = \alpha p \vee \beta q$.

Lower bound.

Suppose B_n is a set of all balancing subsets of n. Then for $\alpha = 1$ there exist q balancing sets. It's easy to see that every set of roots with cardinality equal to $p\alpha$ given by rotations of p^{th} roots of unity is given by the numbers that belong to $\{1, \zeta, \zeta^2, \dots, \zeta^{p-1}\}$. And for every $\alpha \in \{1, 2, \dots, q\}$ the number of balancing sets obtained in this way is equal to $\binom{p}{\alpha}$. Similarly for q. I.e the cardinality $\#B_n \geq \sum_0^p \binom{p}{i} + \sum_0^q \binom{q}{i} = 2^p + 2^q - 2$ (an empty set and a set of cardinality pq is considered twice).

Upper bound.

Obviously, if all the numbers we want to balance have non-negative or non-positive imaginary parts this set won't be balancing. I.e $\#B_n \leq 1 + \sum_{i=2}^{n-2} \binom{n}{i} - \sum_{i=2}^{\left[\frac{n}{2}\right]} \binom{\left[\frac{n}{2}\right]}{i} + 1$.

Case 3. $n = p^k$.

Similarly to 2, $1 + \sum_{i=2}^{n-2} \binom{n}{i} - \sum_{i=2}^{\left[\frac{n}{2}\right]} \binom{\left[\frac{n}{2}\right]}{i} + 1 \geq \#B_n \geq \sum_0^{p^{k-1}} \binom{p^{k-1}}{i} = 2^{p^{k-1}}$

References

[1] On Vanishing Sums of Roots of Unity by T. Y. Lam, K. H. Leung

[2] On Vanishing Sums of Distinct Roots of Unity by Gary Sivek.