4^{th} International Tournament of Young Mathematicians

Belarus

Problem 3. Sums of Powers

Abstract

This problem is about studying the properties of sums of k^{th} powers of complex numbers or integer numbers. For example, we were to prove that if all k^{th} powers are integer, then all coefficients of the polynomial with roots, the set of which coincides with the set $\{z_1, \dots, z_n\}$, are also integers.

We've investigated some (in particular, the first one) points of the problem and we've given the desired answer.

$$(x-z_1)(x-z_2)\dots(x-z_n) = x^n - \sigma_1 x^{n-1} + \dots + (-1)^{n-1}\sigma_{n-1}x + (-1)^n\sigma_n$$

(Viet's formulas)

Let $\alpha \in \mathbb{Q}, m, k \in \mathbb{N}, \{p_1, \dots, p_k\} \in \mathbb{P}$.

Definition 1.0 $A = \{1,2,\dots,n\}$

Definition 1.1 $\alpha \stackrel{\text{def}}{=} m$-good $\Leftrightarrow \alpha \in G_m \Leftrightarrow \alpha \cdot m! \in \mathbb{Z}$

Definition 1.2 $\alpha \stackrel{\text{def}}{=} m$-bad $\Leftrightarrow \alpha \in B_m \Leftrightarrow \alpha \cdot m! \notin \mathbb{Z}$

Definition 1.2' $\alpha \stackrel{\text{def}}{=} \{p_1, \dots, p_k\}, m$-bad $\Leftrightarrow \alpha \in B_m^{\{p_1,\dots,p_k\}} \Leftrightarrow \nu_{p_i}(\alpha \cdot m!) < 0 \ \forall \ p_i$

Claim 1.1 σ_i is i-good $\forall i \in A$

Proof: We'll prove it by induction: $S_1 = \sigma_1 \in G_1$, $\sigma_1 S_1 - S_2 = 2\sigma_2 \Rightarrow \sigma_2 \in G_2$

Suppose $\forall\, i < k: \sigma_i \in X_i$. Observe that $k\sigma_k = \sigma_{k-1}S_1 - \sigma_{k-2}S_2 + \dots + (-1)^{k-1}S_k$ (Newton's identities). After multiplying both sides by $(k-1)!$ all summands on the LHS become (or remain) integer because of our proposition $\Rightarrow k!\,\sigma_k \in \mathbb{Z} \Leftrightarrow \sigma_k \in G_k$ ■

Let $\mathbb{N} \ni i$. Then, according to the Newton Identities:

$$S_{i+n} = S_{i+n-1}\sigma_1 - S_{i+n-2}\sigma_2 + \dots + (-1)^{n+1}S_i\sigma_n \qquad (**)$$

Claim 1.2 σ_n is $(n-1)$-good

Suppose the contrary: σ_n is $\{p_1, \dots, p_k\}, (n-1)$-bad ($p_1, \dots, p_k | n$ because $\sigma_n \in \mathrm{G}_n$)

Then for every p_j after multiplying (**) by $\omega_0 = \frac{n!}{p_j^{\nu_{p_j}(n)}}$ and denoting $\omega_{i>0} = \sigma_i \cdot \omega_0$ we have:

$$(-1)^n S_i\omega_n = S_{i+n-1}\omega_1 - S_{i+n-2}\omega_2 + \dots + (-1)^n S_{i+1}\sigma_{n-1} - \omega_0 S_{i+n}$$

According to our supposition $\omega_n = \frac{A}{p_j^{\alpha}}, (A, p_j) = 1$. All summands on the RHS are integer since $\omega_i \in \mathbb{Z} \ \forall\ i \in A \setminus \{n\} \Rightarrow S_i\omega_n \in \mathbb{Z}\ \forall i \in \mathbb{N} \Rightarrow \nu_{p_j}(S_i) \geq -\nu_{p_j}(\omega_n) = \alpha > 0$.

$$\nu_{p_j}(S_i) > \nu_{p_j}(LHS) = \nu_{p_j}(RHS) \geq \min\{\nu_{p_j}(S_{i+1}), \dots, \nu_{p_j}(S_{i+n})\}$$

I.e. $\forall\ M \in \mathbb{N}\ \exists\ 0 < \alpha \leq n: \nu_{p_j}(S_M) > \nu_{p_j}(S_{M+\alpha}) \Rightarrow \exists\ l \in \mathbb{N}: \nu_{p_j}(S_l) = 0 < \alpha$ – contradiction with our supposition that $\sigma_n$is $p_j, (n-1)$-bad. Similarly, $\sigma_n$isn't $p_i, (n-1)$-bad. ■

Claim 1.3 Suppose $\sigma_g, \sigma_{g+1}, \dots, \sigma_n$ are g-good. Then they are $(g-1)$-good.

If they all $\in G_{g-1}$ we're done. Else, for every prime $p|g$ for which $\exists\ \sigma_j \in B_{g-1}^p$:

After multiplying both sides of (**) by $\omega_0 = \frac{n!}{p^{\nu_p(g)}}$ and defining $\omega_{i>1} = \omega_0\sigma_i$ it's easy to see that all ω_i are of the form $\frac{\Gamma}{p^{\gamma_i}}.\gamma_i = -\nu_p(\omega_i) \stackrel{\text{def}}{=} \deg \omega_i$ $\left(\gamma < \nu_p(g)\ \&\ (\Gamma, p) = 1\right)$.

$$S_{i+n-1}\omega_1 - S_{i+n-2}\omega_2 + \dots + (-1)^n S_{i+1}\sigma_{n-1} + (-1)^{n+1}S_i\omega_n = \omega_0 S_{i+n} \qquad (***)$$

According to our supposition, there exists some non-zero degree γ_τ.

We choose the greatest non-zero degree, say x. After multiplying both sides by p^{x-1} some summands become (remain) integer and the others (with omegas having the greatest degree x) are of the form $\frac{\mathbb{Z}}{p}$. The proof is almost similar to the proof of claim 1.2. Q.E.D.

The proof of the second part of the first point immediately follows from the rational root theorem, i.e., since every root is rational, it's of the form $\frac{a_n}{a_0}$, where a_n is a free term and a_0 is a coefficient at x^n. Since $a_0 = 1$, every root is integer.