4^{th} International Tournament of Young Mathematicians

Team: Belarus

Problem 10. Densities of Natural Subsets

Abstract

This problem is about densities of natural subsets.

In the first part of this work we give a general definition of a density. We consider asymptotic, analytic, logarithmic and other densities and show that these densities are well-defined.

In the second part we recall some well-known relations between the densities and obtain new inequalities for these densities. In particular, we show that if the asymptotic density is equal to zero, then other densities, that we consider, are also equal to zero.

In the third part of our work, we prove the inclusion-exclusion formula for the densities and show that this formula is false for upper densities.

In the fourth part we calculate values of the densities for different natural subsets. We show that the densities of any finite set are equal to zero, the densities of an arithmetic progression with natural members are equal to the inverse value of its difference. We obtain that if a natural sequence grows faster than any arithmetic progression for quite large members, then the densities of such sequence are equal to zero. In particular, the densities of a geometric progression, which denominator is not less than 1, are equal to zero. We calculate the densities of a linear recurrent sequence, in particular, we prove that the densities of the Fibonacci sequence are equal to zero.

The proposed statement of the problem was solved completely.

1 Statement of the Problem

Introduce the following functionals on the set of all subsets of $\mathbb{N}$:

$$\mu_1(E) = \limsup_{n\to+\infty} \frac{\#(E\cap[1,n])}{n}, \text{ for any } E \subseteq \mathbb{N},$$

$$\mu_2(E) = \limsup_{x\to 1-} (1-x)\sum_{n\in E} x^n, \text{ for any } E \subseteq \mathbb{N},$$

$$\mu_3(E) = \limsup_{x\to 1+} (x-1)\sum_{n\in E} \frac{1}{n^x}, \text{ for any } E \subseteq \mathbb{N}.$$

Real numbers $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ will be called *densities* of a set E.
1. Show that the densities μ_1, μ_2 and μ_3 are well-defined.
2. What are the densities $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ of a finite set E?
3. Find the densities $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ of an arithmetic progression

$$E = \{a + nd \mid n \in \mathbb{N}\cup\{0\}\},$$

where a and d are positive integers.
4. Find the densities $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ of a set

$$E = \{[x^n] \mid n \in \mathbb{N}\cup\{0\}\},$$

where $x \geq 1$ is a real number and $[\cdot]$ stands for the integral part.
5. Find or estimate the densities of other subsets of $\mathbb{N}\cup\{0\}$, for instance, of the Fibonacci sequence. Try to construct subsets with "interesting" densities.
6. Is it true that the following equality

$$\mu_1(A\cup B) + \mu_1(A\cap B) = \mu_1(A) + \mu_1(B)$$

holds for any subsets A, $B \in \mathbb{N}$?
7. The same question for μ_2 and μ_3.
8. Introduce and investigate other densities on natural subsets.

2 Introduction

Let $\mathbb{N}$ be the set of natural numbers and $P(\mathbb{N})$ be the set of all natural subsets.

Definition 1. A function $f : X \to [0,1]$, $X \in P(\mathbb{N})$, is called the *density*, if $f(\emptyset) = 0$, $f(\mathbb{N}) = 1$ and for any A, $B \in X$ such that $A \subseteq B$ there holds $f(A) \le f(B)$.

Consider the following functions:

$$\mu_1(E) = \lim_{n\to+\infty} \frac{\#(E \bigcap [1,n])}{n}, \ E \in X_1,$$

$$\mu_2(E) = \lim_{x\to1-} (1-x) \sum_{n\in E} x^n, \ E \in X_2,$$

$$\mu_3(E) = \lim_{x\to1+} (x-1) \sum_{n\in E} \frac{1}{n^x}, \ E \in X_3,$$

$$\mu_4(E) = \lim_{n\to+\infty} \frac{1}{\log n} \sum_{a\in E, a\le n} \frac{1}{a} = \lim_{n\to+\infty} \frac{\sum\limits_{a\in E, a\le n} \frac{1}{a}}{\sum\limits_{a\in \mathbb{N}, a\le n} \frac{1}{a}}, \ E \in X_4,$$

where X_1, X_2, X_3, X_4 are sets of natural subsets such that the limits exist.

It is easy to see that the functions μ_i, $i = 1,2,3,4$, are densities (in the sense of Definition 1). The densities μ_1, μ_3 and μ_4 are known as asymptotic, analytic and logarithmic densities respectively.

If we replace lim by lim sup, then we get upper densities $\bar{\mu}_i$, $i = 1,2,3,4$. It is well known that upper limit of bounded function always exists and is finite. Moreover, $\bar{\mu}_i(\emptyset) = 0$, $\bar{\mu}_i(\mathbb{N}) = 1$ and for any A, $B \in P(\mathbb{N})$ such that $A \subseteq B$ there holds $\bar{\mu}_i(A) \le \bar{\mu}_i(B)$. So the upper densities $\bar{\mu}_i$ are also densities in the sense of Definition 1 and this upper densities are well-defined on $P(\mathbb{N})$.

3 Relations between Densities

Theorem 1. [1]. *The sets X_3 and X_4 coincide and $\mu_3(E) = \mu_4(E)$ for any $E \in X_3$.*

The analogous statement is true for the upper densities $\bar{\mu}_3$ and $\bar{\mu}_4$. [1].

Theorem 2. [2]. *The set X_1 is contained in the set X_3 and for any $E \in X_1$ there holds $\mu_3(E) = \mu_1(E)$. For any set $E \in P(\mathbb{N})$ there holds the inequality $\bar{\mu}_3(E) \le \bar{\mu}_1(E)$.*

It is obviously that $\mu_i(E) = \bar{\mu}_i(E)$ for any $E \in X_i$, $i = 1,2,3,4$. It is also clear that if $\bar{\mu}_i(E) = 0$, then $E \in X_i$ and $\mu_i(E) = 0$ for $i \in \{1,2,3,4\}$.

Theorem 3. *If $\bar{\mu}_1(E) = 0$, then $\bar{\mu}_2(E) = 0$.*

Proof. Let $x = \frac{n-1}{n}$. If $n \to +\infty$ then $x \to 1-0$. Now

$$\bar{\mu}_2(E) = \limsup_{x\to1-} (1-x) \sum_{a\in E} x^a = \limsup_{n\to+\infty} \frac{\sum_{a\in E} (\frac{n-1}{n})^a}{n},$$

$$\bar{\mu}_1(A) = \limsup_{n\to+\infty} \frac{\#\{E \cap [1,n]\}}{n} = \limsup_{n\to+\infty} \frac{\sum\limits_{a\in E, a\le n} 1}{n}.$$

We obtain

$$\bar{\mu}_2(E) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E} (\frac{n-1}{n})^a \le \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E, a\le n} (\frac{n-1}{n})^a + \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E, a>n} (\frac{n-1}{n})^a \le$$

$$\leq \limsup_{n\to+\infty}\frac{1}{n}\sum_{a\in E, a\leq n}1+\limsup_{n\to+\infty}\frac{1}{n}\sum_{a\in E, a>n}(\frac{n-1}{n})^a=\bar{\mu}_1(E)+\limsup_{n\to+\infty}\frac{1}{n}(1-\frac{1}{n})^n\sum_{k\in E_n^1}(\frac{n-1}{n})^k,$$

where $E_n^1=\{a-n|a\in E, a>n\}$.

We need to proof the following Lemma.

Lemma 1. $(1-\frac{1}{n})^n \underset{n\to+\infty}{\uparrow} \frac{1}{e}$.

Since $\lim_{n\to+\infty}(1-\frac{1}{n})^n=\lim_{n\to+\infty}((1-\frac{1}{n})^{-n})^{-1}=e^{-1}$, we only need to show that $f(x)=(1-\frac{1}{x})^x=e^{x\ln(1-\frac{1}{x})}$ is strictly increasing on $(1,+\infty)$. We only need to show that $g(x)=x\ln(1-\frac{1}{x})$ is strictly increasing on $(1,+\infty)$. Let us find the derivative: $g'(x)=\ln(1-\frac{1}{x})+\frac{1}{x-1}$. We need to show $g'(x)>0\ \forall x\in(1,+\infty)$. Let us find the second derivative: $g''(x)=-\frac{1}{(x-1)^2}+\frac{1}{x(x-1)}<0\ \forall x\in(1,+\infty)$. Now $g'(x)>\lim_{n\to+\infty}g'(x)=0\ \forall x\in(1,+\infty)$. Lemma is proved.

By lemma 1 we get $\bar{\mu}_2(E)\leq\frac{1}{e}\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1}(\frac{n-1}{n})^k$.

Now

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1}(\frac{n-1}{n})^k\leq\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k<n}1+\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k\geq n}(\frac{n-1}{n})^k.$$

Consider $\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k<n}1$:

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k<n}1=\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E,n<k<2n}1\leq\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E,k<2n}1.$$

Consequently, $\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k<n}1\leq 2\limsup_{n\to+\infty}\frac{1}{2n}\sum_{k\in E,k<2n}1=2\bar{\mu}_1(E)=0$.

Consider the second summand $\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k\geq n}(\frac{n-1}{n})^k$:

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^1,k\geq n}(\frac{n-1}{n})^k=\limsup_{n\to+\infty}\frac{1}{n}(1-\frac{1}{n})^n\sum_{k\in E_n^2}(\frac{n-1}{n})^k,$$

where $E_n^2=\{a-2n|a\in E, a>2n\}$.

Now by lemma 1 we have $\bar{\mu}_2(E)\leq\frac{1}{e^2}\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^2}(\frac{n-1}{n})^k$.

Analogously we can show that for any natural m:

$$\bar{\mu}_2(E)\leq\frac{1}{e^m}\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^m}(\frac{n-1}{n})^k,$$

where $E_n^m=\{a-mn|a\in E, a>mn\}$.

Since $\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in E_n^m}(\frac{n-1}{n})^k\leq\limsup_{n\to+\infty}\frac{1}{n}\sum_{k\in\mathbb{N}}(\frac{n-1}{n})^k=1$, so $\bar{\mu}_2(E)\leq\frac{1}{e^m}$ for every natural m. Now $\bar{\mu}_2(E)=0$. The theorem is proved.

Theorem 4. *There exists a set E with $\bar{\mu}_1(E)>\bar{\mu}_2(E)>\bar{\mu}_3(E)$.*

Proof. Let E be a subset of natural numbers with first digit 1. It is known [2] that $\bar{\mu}_1(E)=5/9>0.5$ and $\bar{\mu}_3(E)=\log_2 10<0.31$. Let us show that $\bar{\mu}_2(E)\in[0.319,0.447]$. It is easily seen that $\bar{\mu}_2(E)=\limsup_{x\to1-0}\sum_{i=0}x^{10^i}(1-x^{10^i})$. Let $f_i(x)=x^{10^i}(1-x^{10^i})$, $x\in[0,1]$, $i=0,1,\ldots$. Every function $f_i(x)$ is strictly increasing on $[0,a_i]$ and is strictly decreasing on

$[a_i, 1]$, where $a_i = 2^{-10^{-i}}$. Let us show $\sup\limits_{x\in[0,1]} \sum\limits_{i=0}^{\infty} x^{10^i}(1-x^{10^i}) \le 0.447$. It is sufficient to show that $\sup\limits_{x\in[a_{i_0-1},a_{i_0}]} \sum\limits_{i=0}^{\infty} x^{10^i}(1-x^{10^i}) \le 0.447$ for all $i_0 \in \{1,2,\ldots\}$ (we assume that $a_{-1} = 0$). Let us fix $i_0 \in \{1,2,\ldots\}$.

Let us estimate every summand $f_i(x)$, $x \in [a_{i_0-1}, a_{i_0}]$:

1) If $i \le i_0 - 1$, then $\sup\limits_{x\in[a_{i_0-1},a_{i_0}]} f_i(x) = f_i(a_{i_0-1})$;

1) If $i \ge i_0$, then $\sup\limits_{x\in[a_{i_0-1},a_{i_0}]} f_i(x) = f_i(a_{i_0})$.

Let $i > i_0$. Then we have $f_i(a_{i_0}) = (\frac{1}{2})^{10^{i-i_0}}(1-(\frac{1}{2})^{10^{i-i_0}}) \le (\frac{1}{2})^{10^{i-i_0}}$. Now $\sup\limits_{x\in[a_{i_0-1},a_{i_0}]} \sum\limits_{i_0+1}^{\infty} f_i(x) \le \sum\limits_{j=1}^{\infty} (\frac{1}{2})^{10^j}$.

Let $i < i_0 - 1$. Then we obtain $f_i(a_{i_0-1}) = (\frac{1}{2})^{10^{i-i_0+1}}(1-(\frac{1}{2})^{10^{i-i_0+1}}) \le (1-(\frac{1}{2})^{10^{i-i_0+1}})$. Now $\sup\limits_{x\in[a_{i_0-1},a_{i_0}]} \sum\limits_{i=0}^{i_0-2} f_i(x) \le \sum\limits_{j=1}^{\infty}(1-(\frac{1}{2})^{10^{-j}})$.

Consider the function $g(x) = x^{10^{i_0-1}}(1-x^{10^{i_0-1}}) + x^{10^{i_0}}(1-x^{10^{i_0}})$, $x \in [a_{i_0-1}, a_{i_0}]$. Let $\alpha = 10^{i_0-1}$ and $y = x^\alpha$. Now $g(x) = h(y) = y - y^2 + y^{10} - y^{20}$, $y \in [0,1]$. Let us find the global maximum point of $h(y)$. Let us find stationary points: $h'(x) = 1 - 2y + 10y^9 - 20y^{19} = 0$. Let us show that there is only one stationary point y_0 (also it is the global maximum's point of $h(y)$). Function $h''(y) = -2 + 90y^8 - 380y^{18}$ is increasing on $[0, (\frac{2}{19})^{10^{-1}}]$ and $h''(x)$ is decreasing on $[(\frac{2}{19})^{10^{-1}}, 1]$. From $h'(0) = 1 > 0$ and $h'(1) = -11 < 0$ it follows that there is only one solution of $h'(y) = 0$ on $[0,1]$. Using dichotomous search we can find approximation y_1 to y_0 for every $\epsilon > 0$ such that $|y_1 - y_0| < \epsilon$ .Then $|h(y_1) - h(y_0)| < \epsilon \max\limits_{\theta\in[-1,1]} |h(y_0 + \theta\epsilon)| \le 33\epsilon$. Note that $y_0 \approx 0.9140329$ and $\epsilon < 10^{-6}$. Now $h(y_0) \le h(y_1) + 33\epsilon < 0.32$.

Let us estimate $\sum\limits_{j=1}^{\infty}(1-(\frac{1}{2})^{10^{-j}}) + \sum\limits_{j=1}^{\infty}(\frac{1}{2})^{10^j}$.

Let us show that $(1-(\frac{1}{2})^{10^{-j}}) \le 1,125 * 10^{-j}$.

By Taylor's series expansion

$$(1+x)^\mu = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \ldots + \frac{\mu(\mu-1)\ldots(\mu-n+1)}{n!}x^n + R_n(x),$$

where $|R_n(x)| \le \frac{\sup\limits_{\theta\in[0,1]} |f^{(n)}(\theta x)|}{n!} x^n$, $f(x) = (1+x)^\mu$.

Now $1-(\frac{1}{2})^{10^{-j}} = 1-(1-0.5)^{10^{-j}} = 0.5^{10^{-j}} - \frac{10^{-j}(10^{-j}-1)}{2!}0.25 + R_2(x) \le 0.625 * 10^{-j} + R_2(x)$. $R_2(x) \le \frac{10^{-j}(1-10^{-j})}{2!} \le 0.5 * 10^{-j}$. So $1-(\frac{1}{2})^{10^{-j}} \le 1.125 * 10^{-j}$.

Now $\sum\limits_{j=1}^{\infty}(1-(\frac{1}{2})^{10^{-j}}) \le 1.125 * 10^{-1}\frac{1}{1-10^{-1}} = 0.125$.

We have $\sum\limits_{j=1}^{\infty}(\frac{1}{2})^{10^j} \le \sum\limits_{j=10}^{\infty}(\frac{1}{2})^j = \frac{1}{1024}\frac{1}{1-0.5} = \frac{1}{512} < 0.002$.

Finally $\sup\limits_{x\in[a_{i_0-1},a_{i_0}]} \sum\limits_{i=0}^{\infty} x^{10^i}(1-x^{10^i}) \le 0.32 + 0.125 + 0.002 = 0.447$.

Let us show that $\limsup\limits_{x\to 1-0} \sum\limits_{i=0}^{\infty} x^{10^i}(1-x^{10^i}) \ge 0.319$. Consider the sequence $\{x_n\}$, $x_n = y_0^{10^{-n}}$ (y_0 the same as was defined below). Now $\sum\limits_{i=0}^{\infty} x_n^{10^i}(1-x_n^{10^i}) \ge h(y_1) > 0.319$ for every natural n. So $\bar{\mu}_2(E) > 0.319$. The theorem is proved.

Theorem 5. There holds $\bar{\mu}_1(E) + \frac{1}{e} \ge \bar{\mu}_2(E)$ for any $E \subseteq \mathbb{N}$.

Proof. We obtain

$$\bar{\mu}_2(E) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E} (\frac{n-1}{n})^a \le \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E, a\le n} (\frac{n-1}{n})^a + \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E, a>n} (\frac{n-1}{n})^a \le$$

$$\le \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E, a\le n} 1 + \limsup_{n\to+\infty} \frac{1}{n} \sum_{a\in E, a>n} (\frac{n-1}{n})^a \le \bar{\mu}_1(E) + \limsup_{n\to+\infty} \frac{1}{n} \sum_{a=n+1}^{\infty} (\frac{n-1}{n})^a =$$

$$= \bar{\mu}_1(E) + \lim_{n\to+\infty} (\frac{n-1}{n})^{n+1} = \bar{\mu}_1(E) + \frac{1}{e}.$$

Theorem 5 is proved.

4 Inclusion-Exclusion Formula for Densities

Theorem 6. *Let $i \in \{1, 2, 3, 4\}$. Suppose that sets A, B and $A \cap B$ are contained in the set X_i, then the set $A \cup B$ is also contained in X_i and there holds the equality*

$$\mu_i(A \cup B) = \mu_i(A) + \mu_i(B) - \mu_i(A \cap B).$$

Proof. Let A, B, $A \cap B \in X_i$. Consider the case: $i = 1$. Since

$$\#((A \cup B) \cap [1, n]) = \#(A \cap [1, n]) + \#(B \cap [1, n]) - \#(A \cap B \cap [1, n])$$

and the limits

$$\lim_{n\to+\infty} \frac{\#(A \bigcap [1, n])}{n}, \lim_{n\to+\infty} \frac{\#(B \bigcap [1, n])}{n}, \lim_{n\to+\infty} \frac{\#(A \cap B \bigcap [1, n])}{n},$$

exist, so the limit

$$\lim_{n\to+\infty} \frac{\#((A \cup B) \bigcap [1, n])}{n}$$

is also exists, and $\mu_1(A \cup B) = \mu_1(A) + \mu_1(B) - \mu_1(A \cap B)$.

Introduce the indicator function for elements of a set: let $1_X(x) = 1$, if $x \in X$ and $1_X(x) = 0$, if $x \notin X$. It is clear that

$$\mu_2(A) + \mu_2(B) - \mu_2(A \cap B) = \lim_{x\to 1-} \left(\sum_{n=1}^{\infty} 1_A(n)x^n + \sum_{n=1}^{\infty} 1_B(n)x^n - \sum_{n=1}^{\infty} 1_{A\cap B}(n)x^n \right),$$

$$\mu_3(A) + \mu_3(B) - \mu_3(A \cap B) = \lim_{x\to 1+} \left(\sum_{n=1}^{\infty} 1_A(n)\frac{1}{n^x} + \sum_{n=1}^{\infty} 1_B(n)\frac{1}{n^x} - \sum_{n=1}^{\infty} 1_{A\cap B}(n)\frac{1}{n^x} \right).$$

As $1_A(n) + 1_B(n) - 1_{A\cap B}(n) = 1_{A\cup B}(n)$ and the series $\sum_{n=1}^{\infty} 1_A(n)x^n$, $\sum_{n=1}^{\infty} 1_B(n)x^n$, $\sum_{n=1}^{\infty} 1_{A\cap B}(n)x^n$ converge absolutely for any fixed $x \in (0, 1)$, the series $\sum_{n=1}^{\infty} 1_A(n)\frac{1}{n^x}$, $\sum_{n=1}^{\infty} 1_B(n)\frac{1}{n^x}$, $\sum_{n=1}^{\infty} 1_{A\cap B}(n)\frac{1}{n^x}$ converge absolutely for any fixed $x > 1$, so we get

$$\mu_2(A \cup B) = \mu_2(A) + \mu_2(B) - \mu_2(A \cap B),$$

$$\mu_3(A \cup B) = \mu_3(A) + \mu_3(B) - \mu_3(A \cap B).$$

The statement of the theorem for μ_4 follows from Theorem 1. This theorem is proved.

Theorem 7. *For any $i \in \{1,2,3,4\}$ there exist sets A and B such that*

$$\bar{\mu}_i(A \cup B) < \bar{\mu}_i(A) + \bar{\mu}_i(B) - \bar{\mu}_i(A \cap B).$$

Proof.

1) Consider the sequence

$$A = \{1,\ 3, 4,\ 7, 8, 9, 10,\ 15, 16, 17, 18, 19, 20, 21, 22,\ 31, \ldots\}$$

constructed from the sequence $\mathbb{N} = \{1, 2, 3...\}$ by the way: the first number we include, the second we exclude (1st period), then two numbers we include , two numbers we exclude (2st period),..., 2^i numbers we include, 2^i numbers we exclude (ith period)...

We see that the superior of $\frac{\#\{A \cap [1,n]\}}{n}$ is achieved in the middle of each period, i.e. 1,4 ,10,22 ... It has a number in the sequence $n_i = 2(\sum_{j=0}^{i-1} 2^j) + 2^i = 2^{i+1} - 2 + 2^i$. And $b_i = \#\{A \cap [1, n_i]\} = \sum_{j=0}^{i-1} 2^j + 2^i = 2^{i+1} - 1$.

$$\bar{\mu}_1(A) = \limsup_{i \to +\infty} \frac{b_i}{n_i} = \lim_{i \to +\infty} \frac{2^{i+1} - 1}{2^{i+1} + 2^i - 2} = 2/3.$$

Let $B = \mathbb{N} \setminus A$. Note that superiors are achieved after each period, it means

$$\bar{\mu}_1(B) = \limsup_{n \to +\infty} \frac{\#\{B \cap [1,n]\}}{n} = \lim_{m \to +\infty} \frac{\sum_{i=0}^{m} 2^i}{2\sum_{i=0}^{m} 2^i} = 1/2.$$

Hence

$$\bar{\mu}_1(A) + \bar{\mu}_1(B) = 7/6 > \bar{\mu}_1(A \cap B) + \bar{\mu}_1(A \cup B) = 1.$$

2) Let E be a subset of natural numbers with first digit 1. Let us divide E into two subsets. In first subset A we put all numbers from E with even number of digits and in the second B we put all numbers from E with odd number of digits.

Consider the sequences $\{a_i\} = 2^{-10^{-(2i-1)}}$. We have that if $i \to +\infty$ then $a_i \to 1-$. Since $\sum\limits_{i=1}^{\infty} a_n^{10^{2i-1}}(1 - a_n^{10^{2i-1}}) \geq 0.25 \forall n \in \mathbb{N}$, so $\bar{\mu}_2(A) \geq 0.25$. By analogy $\bar{\mu}_2(B) \geq 0.25$. Now $\bar{\mu}_2(A) + \bar{\mu}_2(B) \geq 0.5 > 0.447 \geq \bar{\mu}_2(A \cap B) + \bar{\mu}_2(A \cup B)$.

3) Let us divide $\mathbb{N}$ into two subsets by the following rule. First $2^{10^{10^0}}$ numbers belong to the first subset A the next $2^{10^{10^1}}$ belong to the second subset B the next $2^{10^{10^2}}$ to A ....

Let us find $\overline{\mu_4}$ from A and B.

Consider the sequence a_i in $\mathbb{N}$ such that $a_i \in A$ for all i and $a_i + 1$ does not in A for all i. Consider the sequence b_i in Z_+ such that $b_1 = 0$, $b_i + 1 \in A$ for all i and b_i does not in A for all i. (We mean that all numbers satisfying conditions described below belong to that sequences).

By the way of constructing A we have that $a_i - b_i = 2^{10^{10^{2(i-1)}}}$ and $b_i = \sum\limits_{j=0}^{2i-3} 2^{10^{10^j}} \leq 2 * 2^{10^{10^{2(i-1)-1}}}$ for $i > 1$. Consequently

$\bar{\mu}_4(A) \geq \limsup_{i\to+\infty} \frac{\sum\limits_{a \in A, a \leq a_i} 1/a}{\sum\limits_{a \leq a_i} 1/a} \geq \limsup\limits_{i \to +\infty} \frac{\sum\limits_{a \leq a_i} 1/a - \sum\limits_{a < b_i} 1/a}{\sum\limits_{a \leq a_i} 1/a} = \limsup\limits_{i \to +\infty}(1 - \frac{\sum\limits_{a < b_i} 1/a}{\sum\limits_{a \leq a_i} 1/a}) \geq \lim\limits_{i \to +\infty}(1 - \frac{\ln(2*2^{10^{10^{2(i-1)-1}}})}{\ln(2^{10^{10^{2(i-1)}}})}) = 1.$

So $\bar{\mu}_4(A) = 1$. By analogy we have that $\bar{\mu}_4(B) = 1$.

Now $\bar{\mu}_4(A) + \bar{\mu}_4(B) = 2 > \bar{\mu}_4(A \cap B) + \bar{\mu}_4(A \cup B) = 1$.

4) For the density $\bar{\mu}_3$ we can get the same sets A and B, as for the density $\bar{\mu}_4$. The theorem is proved.

5 Densities of Particular Sets

5.1 Finite Sets

Theorem 8. *Let E be a finite subset of $\mathbb{N}$, then $\mu_i(E) = 0$ for any $i \in \{1, 2, 3, 4\}$.*

Proof. The value $\#\{E \cap [1, ..., n]\}$ is upper bounded by a constant L that doesn't depend on n, hence $\#\{E \cap [1, n]\}/n \to 0$ as $n \to +\infty$. So $\mu_1(E) = 0$. Theorems 1, 2 and 3 imply the truth of the equalities $\mu_i(E) = 0$ for $i \in \{2, 3, 4\}$. The theorem is proved.

Theorem 9. *Let $i \in \{1, 2, 3, 4\}$, $A \in X_i$ and E be a finite set, then $A \cup E \in X_i$ and $\mu_i(A \cup E) = \mu_i(A)$.*

Proof follows from theorem 6 and 8.

5.2 Arithmetic Progression

Let $E = \{dn + a | n \in \mathbb{N} \cup \{0\}\}$, where a, $d \in \mathbb{N}$.

Theorem 10. *There holds $\mu_i(E) = 1/d$ for any $i \in \{1, 2, 3, 4\}$.*

Proof.

1) Denote

$$a_n = \frac{\#(E \cap [1, n])}{n}.$$

It is easy to see that $\frac{(n-a)/d}{n} \le a_n \le \frac{(n-a)/d+1}{n}$. Note that $\frac{(n-a)/d}{n} \to 1/d$ and $\frac{(n-a)/d+1}{n} \to 1/d$ as $n \to +\infty$. We get

$$\lim_{n \to +\infty} \frac{\#(E \cap [1, n])}{n} = 1/d.$$

2) We obtain

$$\lim_{x \to 1-0}(1 - x) \sum_{n \in E} x^n = \lim_{x \to 1-0}(1 - x) \sum_{i=0}^{\infty} x^{a+id} = \lim_{x \to 1-0}(1 - x)x^a \frac{1}{1-x^d} = \lim_{x \to 1-0} x^a \frac{1}{\sum_{i=0}^{d-1} x^i} = 1/d.$$

3) We need to prove two lemmas.

Lemma 2. Let $A \in X_3$, then for any natural number m the set $m \cdot A := \{ma \mid a \in A\}$ is also contained in X_3 and $\mu_3(m \cdot A) = \mu_3(A)/m$.

Proof. We have

$$\mu_3(m \cdot A) = \lim_{x \to 1+}(x - 1) \sum_{n \in A} \frac{1}{(mn)^x} = \lim_{x \to 1+}(x - 1)\frac{1}{m^x} \sum_{n \in A} \frac{1}{n^x} = \mu_3(A)/m.$$

Lemma is proved.

Lemma 3. Let $A \in X_3$, then for any natural number m the set $m + A := \{m + a \mid a \in A\}$ is also contained in X_3 and $\mu_3(m + A) = \mu_3(A)$.

Proof. It is sufficient to prove the lemma for the case $m = 1$. Let $A = \{a_1, a_2, ..., a_i, ...\}$, where $a_i < a_j$ for $i < j$. Denote $A_N = A \setminus \{a_1, ..., a_{N-1}\}$ for any natural number N. It is evident that the sets A_n have the same density μ_3 as the set A. Also $\mu_3(1 + A) = \mu_3(1 + A_N)$ for all $N \in \mathbb{N}$.

It is clear that for any $x > 1$ there hold the inequalities

$$(x - 1) \sum_{n \in A_N} \frac{1}{\left(\frac{a_N+1}{a_N} n\right)^x} \le (x - 1) \sum_{n \in A_N} \frac{1}{(n + 1)^x} \le (x - 1) \sum_{n \in A_N} \frac{1}{n^x}.$$

We have

$$\lim_{x \to 1+}(x - 1) \sum_{n \in A_N} \frac{1}{n^x} = \mu_3(A_N) = \mu_3(A).$$

Applying lemma 2, we get

$$\lim_{x\to 1+}(x-1)\sum_{n\in A_N}\frac{1}{\left(\frac{a_N+1}{a_N}n\right)^x}=\frac{a_N}{a_N+1}\mu_3(A_N)=\frac{a_N}{a_N+1}\mu_3(A).$$

We fixed an arbitrary $\varepsilon>0$. There exists a natural number N such that $\frac{a_N}{a_N+1}\mu_3(A)>\mu_3(A)-\varepsilon$.

Since

$$\lim_{x\to 1+}(x-1)\sum_{n\in A_N}\frac{1}{(n+1)^x}=\mu_3(1+A_N)=\mu_3(1+A),$$

so we obtain

$$\mu_3(A)-\varepsilon\le\mu_3(1+A)\le\mu_3(A).$$

Consequently, $\mu_3(m+A)=\mu_3(A)$. So the lemma is proved.

It follows from lemmas 2 and 3 that $\mu_3(E)=\mu_3(d\cdot\mathbb{N})=\mu_3(\mathbb{N})/d=1/d$.

By theorem 1 we have $\mu_4(E)=\mu_3(E)=1/d$. The theorem is proved.

5.3 Geometric Progression

Theorem 11. *Let $A=\{a_i\}_{i\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that for any natural number k there exist natural numbers $\alpha=\alpha(k)$ and $i_0=i_0(k)$ such that there holds the inequality $a_i>\alpha+ki$ for any $i\ge i_0$. Then $\mu_i(A)=0$ for any $i\in\{1,2,3,4\}$.*

Proof. It is sufficient to prove that $\bar\mu_1(A)=0$. Suppose the contrary, then $\bar\mu_1(A)=d>0$. It means that for an arbitrary $\varepsilon>0$ there exist a strictly increasing sequence of natural numbers $\{n_m\}_{m\in\mathbb{N}}$ and a natural number $m_0=m_0(\varepsilon)$ such that for any $m\ge m_0$ there holds

$$\sum_{a\in A,a\le n_m}1\ge(d-\varepsilon)n_m.$$

There exist natural numbers α and i_0 such that $a_i>\alpha+\frac{2}{d}i$ for any $i\ge i_0$.

Let us estimate the sum $\sum\limits_{a\in A,a\le n_m}1$, where $m\ge m_0$. Let $j_m=\max\{i|a_i\in A,a_i\le n_m\}$. If $j_m\le i_0$, then $\sum\limits_{a\in A,a\le n_m}1\le i_0$, else $\alpha+\frac{2}{d}j_m<a_{j_m}\le n_m$. Hence, in the case $j_m>i_0$ we have $j_m<\frac{d}{2}n_m$ and $\sum\limits_{a\in A,a\le n_m}1<\frac{d}{2}n_m$. We prove that there holds the inequality

$$\max\left\{i_0,\frac{d}{2}n_m\right\}>(d-\varepsilon)n_m$$

for all $m\ge m_0$. But this inequality is impossible for all quite large m, if $0<\varepsilon<\frac{1}{2}$. The theorem is proved.

Corollary 1. *Let $A=\{a_i\}_{i\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that there exist positive real constants r, $n>1$, i_0 such that the inequality $a_i>ri^n$ holds for any $i\ge i_0$. Then $\mu_i(A)=0$ for any $i\in\{1,2,3,4\}$.*

Proof. Let us fix an arbitrary natural k. It is sufficiently to show that there exist natural numbers $\alpha=\alpha(k)$ and $j_0=j_0(k)$ such that the inequality $ri^n\ge\alpha+ki$ holds for any $i\ge j_0$. We can choose $\alpha=1$, $j_0=\max\{i_0,(2k/r)^{1/(n-1)}\}$. The corollary is proved.

Corollary 2. *Let $C=\{[x^n]|n\in\mathbb{N}\cup 0\}$, where $x\ge 1$. Then $\mu_1(C)=\mu_2(C)=\mu_3(C)=\mu_4(C)=0$.*

Proof. If $x=1$ then C is finite. So for $x=1$ the corollary is proved. Suppose that $x>1$.

It is obviously that there exists $j>0$ such that $\forall i\ge j$ the inequality $[x^i]>i^2$ holds.

Let $C=\{[x^{c_i}]\}_{i\in\mathbb{N}}$, where $\{c_i\}_{i\in\mathbb{N}}$ is an strictly increasing sequence. The inequalities $[x^{c_i}]\ge[x^i]>i^2$ $\forall i\ge j$ and corollary 1 immediately imply the desired statement.

5.4 Linear Recurrent Sequence

Let $\mathbb{R}_+ = [0, +\infty)$. We consider an arbitrary linear recurrent sequence

$$x_n = ax_{n-1} + bx_{n-2},\ n \geq 2,$$

with fixed initial values x_0, $x_1 \in \mathbb{R}_+$ and fixed a, $b \in \mathbb{R}_+$. Let us construct the following set: $X = \{\lceil x_n \rceil \mid n \in \mathbb{N} \cup \{0\}\}$, where $\lceil x \rceil = \max\{[x], 1\}$.

Denote by ν_1 and ν_2 the roots of the equation

$$\nu^2 - a\nu - b = 0.$$

Theorem 12. *For any fixed a, b, x_0, $x_1 \in \mathbb{R}_+$ there holds $\mu_i(X) = 0$ for $i \in \{1, 2, 3, 4\}$.*

Proof. In the trivial case $a = b = 0$ the theorem is true. Suppose that $a > 0$ or $b > 0$, then $\nu_1 \neq \nu_2$. Consequently, for any $n \in \mathbb{N} \cup \{0\}$ there holds the equality:

$$x_n = C_1\nu_1^n + C_2\nu_2^n,$$

where C_1, $C_2 \in \mathbb{R}$.

If $|\nu_1| \leq |\nu_2| \leq 1$, then $x_n \leq |C_1| + |C_2|$. In this case the set X is finite and $\mu_i(X) = 0$.

Suppose that $|\nu_1| < |\nu_2|$ and $|\nu_2| > 1$. If $C_2 \neq 0$, then we obtain

$$x_n \geq |C_2||\nu_2|^n \left(1- \mid C_1/C_2 \mid\mid \nu_1/\nu_2 \mid^n\right) \geq \frac{1}{2}|C_2||\nu_2|^n$$

for all quite large n. Corollary 1 implies that $\mu_i(X) = 0$. If $C_2 = 0$, then we have two cases: 1) $C_1 = 0$, then $x_n \equiv 0$; 2) $C_1 \neq 0$, then x_n either uniformly bounded or $x_n \geq |C_1||\nu_1|^n$, where $|\nu_1| > 1$. In all these cases we have $\mu_i(X) = 0$.

It remains to consider the case, when $|\nu_1| = |\nu_2| > 1$. Since

$$\nu_{1,2} = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

and $\nu_1^2 = \nu_2^2$, so $a\sqrt{a^2 + 4b} = 0$, that implies the equality $a = 0$. In this case we get $x_n = bx_{n-2}$ $\forall n \geq 2$. If $\min\{x_0, x_1\} = \alpha > 0$, then $x_n \geq \alpha b^{(n-1)/2}$. As $b = |\nu_1|^2 = |\nu_2|^2 > 1$, so by theorem 11 we have $\mu_i(X) = 0$. If one from the numbers x_0 and x_1 is equal to 0, then $X = \{\lceil \gamma b^n \rceil \mid n \in \mathbb{N} \cup \{0\}\}$, where $\gamma = \max\{x_0, x_1\}$. If $\gamma = 0$, then X is a finite set, else, applying corollary 1, we obtain that $\mu_i(X) = 0$. The theorem is proved.

Corollary 3. *Let F be the Fibonacci sequence, then $\mu_i(F) = 0$ for any $i \in \{1, 2, 3, 4\}$.*

6 References

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