4^{th} **International Tournament of Young Mathematicians**

Team: Belarus

Problem 1. Generalizing Perfectness

Abstract

This problem is about generalized perfect numbers. We call a natural number n f-perfect for some arithmetic function f, if $f(n)$ is equal to the sum of $f(d)$ over all proper divisors d of the number n. The following results are obtained in this work:

We find all f-perfect numbers for functions for arithmetic functions $f(a)$ satisfying $f(ab) = f(a)+f(b)$ and the following functions: $(-1)^n$, i^n, C_{2012}^n and Jordan totient function. We find all $\tau+t$-perfect numbers for $t \in \mathbb{N} \cup \{0, -1, -2, -2^2, \dots\}$ where $\tau(n)$ is the number of divisors of n. Also we find sufficient condition on n to be f-perfect for $f(x) = ax + b$.

We call a pair of natural numbers (m, n) amicable if n is equal to the sum of d over all proper divisors d of the number m and m is equal to the sum of d over all proper divisors d of the number n. We generalize this notion and consider f-amicable pairs for some arithmetic function f. There are found all f-amicable pairs, when f is the Jordan function or $f(n) = (-1)^n$.

Also we generalize perfectness on other sets. For example we find all perfect numbers in the ring of residues. We constructed the set closed under two operations such that all numbers of it are perfect.

Notation

1. $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{P}$ are the sets of complex, real, integer, nonnegative integer, natural, prime numbers.
2. We adopt the usual convention of assigning the value 0 to any empty sum and the value 1 to the any empty problem.
3. Z_n is the ring of residues modulo n and $\bar{a}$ is the residue of a modulo n.

Introduction

Any function $f : \mathbb{N} \to \mathbb{C}$ is called *arithmetic*. Recall that a natural number $n \geq 1$ is said to be *perfect* if it is equal to the sum of its positive proper divisors (that is all divisors except n). Examples of perfect numbers are 6, 28 and 496. Euclid proved that if k is a natural number such that $2^{k+1} - 1$ is prime, then $n = 2^k(2^{k+1} - 1)$ is perfect.

Generalizing this notion, we say that a natural number $n \in \mathbb{N}$ is *f-perfect* for some arithmetic function f if

$$f(n) = \sum_{d|n, 1 \leq d \leq n-1} f(d).$$

Thus, n is perfect if and only if n is f-perfect for $f(n) = n$. For another example, note that a natural number $n \geq 1$ is f-perfect for the constant function $f(n) = 1$ if and only if n is a prime.

Note that iff n is f-perfect then

$$\sum_{d|n} f(d) = 2f(n).$$

Question 1

Let $\tau(n)$ denote the number of positive divisors of n (including n).

a) Prove that a natural number $n \geq 1$ is τ-perfect if and only if n is a square of a prime.

b) Find all f-perfect natural numbers $n \geq 1$ for the function $f(n) = \tau(n) - 1$. For as many values $k \in \mathbb{Z}$ as possible, find all f-perfect natural numbers $n \geq 1$ for $f(n) = \tau(n) + k$.

Solution. We need the following lemma:

Lemma 1.1. *Let* $n = \prod\limits_{p_i \in \mathbb{P}} p_i^{\alpha_i}$. *Then* $\sum\limits_{d|n} (\tau(d) + t) = \prod\limits_{i} \frac{(\alpha_i+1)(\alpha_i+2)}{2} + t\tau(n)$.

Proof. It is obvious that $\sum\limits_{d|n} t = t\tau(n)$. Since $\tau(n)$ is multiplicative and for a prime p we have $\tau(p^i) = i + 1$, so

$$\sum_{d|n}\tau(d)=\prod_i(\sum_{j=0}^{\alpha_i}\tau(p_i^j))=\prod_i\frac{(\alpha_i+1)(\alpha_i+2)}{2}.$$

Lemma 1 is proved.

Theorem 1.1. *There is no $(\tau+t)$-perfect number for $t\geq 1$.*

Proof. Assume that n is $(\tau+t)$-perfect. We have that if $t\geq 2$ then

$$\sum_{d|n}\tau(d)=\prod_i\frac{(\alpha_i+1)(\alpha_i+2)}{2}=(2-t)\tau(n)+2t. \tag{1}$$

Consider $t\geq 2$. Note that $n=1$ is not a solution of (1). It means that $\tau(n)\geq 2$. So $(2-t)\tau(n)\leq 4-2t$ or $(2-t)\tau(n)+2t\leq 4$ and $\prod_i\frac{(\alpha_i+1)(\alpha_i+2)}{2}\geq 3$. Since $\tau(n)\geq 2$, so $\prod_i\frac{(\alpha_i+1)(\alpha_i+2)}{2}$ have an nonunit odd divisor. Now $(2-t)\tau(n)+2t=3$. Since

$$\prod_i\frac{(\alpha_i+1)(\alpha_i+2)}{2}=\tau(n)\prod_i(1+\frac{\alpha_i}{2})=3,$$

we have $\tau(n)=2$ and $(2-t)\tau(n)+2t=4$. Now $4=3$. It is a contradiction. So there is no $(\tau+t)$-perfect number for $t\geq 2$.

Let $t=1$. We have

$$\prod_i(1+\frac{\alpha_i}{2})=1+\frac{2}{\tau(n)}. \tag{2}$$

It is easy to check that (2) has no solution for $\tau(n)=1$ and $\tau(n)=2$. When $\tau(n)>2$ the left hand side (LHS) is more than 2 and the right hand side (RHS) is less than 2. Theorem 1.1 is proved.

Proposition 1.1. *All τ-perfect number are $n=p^2$ where p is a prime.*

Proof. Let $t=0$. We have

$$\tau(n)\prod_{p_i}(1+\frac{\alpha_i}{2})=2\tau(n) \tag{3}$$

or $\prod_{p_i}(1+\frac{\alpha_i}{2})=2$. Note that $n=p^2$ is solution of (3). If n has 2 or more prime divisors or one of α_i is more than 2 then $\prod_{p_i}(1+\frac{\alpha_i}{2})>2$. So $n=p^2$ where p is a prime. Proposition 1.1 is proved.

Theorem 1.2. *If n is $(\tau-2^k)$-perfect number then $k=0$ and n is a cube of a prime or 1.*

Proof. Let $d(n)$ be the number of prime divisors of n and $t=-2^k$. We have:

$$2^{d(n)}\tau(n)\prod_{i=1}^{d(n)}(1+\frac{\alpha_i}{2}) = 2^{d(n)}(2+2^k)\tau(n) - 2^{d(n)+1+k}. \qquad (4)$$

Since LHS is divisible by $\tau(n)$, so $2^{d(n)+1+k}$ is divisible by $\tau(n)$. It means that $\tau(n)$ is a power of 2. Now $2^{d(n)}\prod_{i=1}^{d(n)}(1+\frac{\alpha_i}{2})$ is odd. We have two cases:

1) $2^{d(n)}(2+2^k)$ is odd and $2^{d(n)+1+k}/\tau(n)$ is even. It means that $d(n) = k = 0$. So we have that $n = 1$, $k = 0$ is solution of (4).

2) $2^{d(n)}(2+2^k)$ is even and $2^{d(n)+1+k}/\tau(n)$ is odd. It means that $\tau(n) = 2^{d(n)+1+k}$. Now

$$2^{d(n)}\prod_{i=1}^{d(n)}(1+\frac{\alpha_i}{2}) = 2^{d(n)+1} - 1 + \frac{\tau(n)}{2}.$$

Since $2^{d(n)}\prod_{i=1}^{d(n)}(1+\frac{\alpha_i}{2}) > \tau(n)$, $2^{d(n)+1} - 1 > \tau(n)/2$. Since $\tau(n)$ is a power of 2, we have $2^{d(n)+1+k} = \tau(n) \le 2^{d(n)+1}$. Since k is nonnegative, so $\tau(n) = 2^{d(n)+1}$ and $t = -1$. From $\tau(n) = 2^{d(n)+1}$ it follows that one of $\{\alpha_i\}$ is 3 and the others are 1. We have $5 \cdot 3^{d(n)-1} = 3 \cdot 2^{d(n)} - 1$. If $d(n) > 1$ then LHS is divisible by 3 and RHS is not divisible by 3, a contradiction. So $d(n) = 1$. It means that n is a cube of a prime. Note that $n = p^3$, $p \in \mathbb{P}$ is solution of (4). Theorem 1.2 is proved.

Question 2.

Find all f-perfect numbers n, where $f(n) = \varphi(n)$ is Euler's totient function.

Solution. Let $J_a(n) = n^a \prod_{p|n, p\in\mathbb{P}}(1-\frac{1}{p^a})$ be Jordan totient function.

Theorem 2.1. *2 is the only J_1-perfect (φ-perfect) number. And there is no J_a-perfect number for $a > 1$.*

Proof. It is well known [1, p. 91] that $\sum_{d|n} J_a(d) = n^a$.

So we must solve $n^a = 2J_a(n)$ or $\prod_{p|n,p\in\mathbb{P}}(1-\frac{1}{p^a}) = 1/2$.

Note numerator of LHS is divisible by 2^k ($k = \#(p : p|n, p \in \mathbb{P}, p \neq 2)$ and denominator is divisible by 2^a if n is even and is not divisible by 2 if n is odd. It means that n is divisible by not more then $a-1$ odd primes. Note that $\prod_{p|n,p\in\mathbb{P}}(1-\frac{1}{p^a})$ is not less then $(1-\frac{1}{2^a})^{1+(a-1)} \ge 1 - \frac{a}{2^a}$ (the last is obtained by Bernoulli inequality). We have that$1 - \frac{a}{2^a} = 1/2$ for $a = 1$ (and it is easy to check that 2 is J_1-perfect) and $1 - \frac{a}{2^a} > 1/2$ for $a > 1$. It means that there is no J_a-perfect numbers for $a > 1$ and 2 is the only J_1-perfect (φ-perfect) number. Theorem 2.1 is proved.

Question 3

a) Prove that if k is a natural number such that $2^{k+1} - 2k - 1$ is a prime, then $n = 2^k(2^{k+1} - 2k - 1)$ is f-perfect for $f(n) = n - 1$.

b) Find similar sufficient conditions for f-perfectness for other polynomial functions of degree 1 such that $f(n) = n - 2$ or $f(n) = n + 1$.

Solution. Let us consider the general case $f(n) = an + b$ where $a, b \in \mathbb{N} \cup \{0\}$ and $a \neq 0$.

Theorem 3.1 *If* $p = (2^{k+1} - 1 + 2bk/a)$ *is a prime, then* $n = 2^k p$ *is* f*-perfect for* $f(n) = an + b$.

Proof. If n is f-perfect then $\sum\limits_{d|n} f(d) = a(p(1 + 2 + \cdots + 2^k) + (1 + 2 + \cdots + 2^k)) + 2b(k + 1) = a(2^{k+1}p + 2^{k+1} - p - 1) + 2b(k + 1) = 2a2^k p + 2b$ or $p = 2^{k+1} - 1 + 2bk/a$. Theorem 3.1 is proved.

Corollary 3.1 (Euclid). *If* $p = (2^{k+1} - 1)$ *is a prime, then* $n = 2^k p$ *is perfect.*

Corollary 3.2. *If* $2^{k+1} - 2k - 1$ *is a prime then* $n = 2^k(2^{k+1} - 2k - 1)$ *is* f*-perfect for* $f(n) = n - 1$.

Question 4

Let $f(n) = \ln(n)$. Find all f-perfect numbers n.

Solution. It is known that $\ln(ab) = \ln(a) + \ln(b)$. That is why we will consider the general case: function f satisfying $f(ab) = f(a) + f(b)$.

Theorem 4.1 *Let* $f(n)$ *be function satisfying* $f(ab) = f(a) + f(b)$. *Then all* f*-perfect numbers are 1,* p^3*,* pq *where* p *and* q *are a primes and* x *such that* $f(x) = 0$.

Proof. It is obvious that $f(1) = 0$. Let us show $\sum\limits_{d|n} f(d) = \frac{\tau(n)}{2} f(n)$. Since $f(ab) = f(a) + f(b)$, $f(n) = f(n/d) + f(d)$ for all $d|n$. If n is not a perfect square then $n/d \neq d$ for all $d|n$. We have $\tau(n)/2$ pairs $(n/d, d)$. That is why $\sum\limits_{d|n} f(d) = \frac{\tau(n)}{2} f(n)$. If n is a perfect square then we have $\frac{\tau(n)-1}{2}$ pairs $(n/d, d)$ such that $d|n$ and $n/d \neq d$. Also for $\sqrt{n}$ we have: $f(\sqrt{n}) = 1/2 f(n)$ That is why $\sum\limits_{d|n} f(d) = \frac{\tau(n)}{2} f(n)$.

Now $\sum\limits_{d|n} f(d) = \frac{\tau(n)}{2} f(n) = 2f(n)$. We have two cases:

I. $\tau(n) = 4$. It means that $n = pq$ or $n = p^3$ where p and q are primes.

II. $f(n) = 0$. It means that n is also solution. Theorem 4.1 is proved.

Note, that if $f : \mathbb{R} \to \mathbb{R}$ and $f(n)$ is continuous then $f(x) = \log_a x$ (Equation of Cauchy). But there are a lot of other such arithmetic functions.

For example let π be a subset of the set of all primes. Let $n = \prod_{p_i \in \mathbb{P}} p_i^{\alpha_i}$. $f(n) = \sum_{p_i | n, p_i \in \pi} \alpha_i$.

Corollary 4.1. *All* ln*-perfect numbers are 1, p^3, pq where p and q are a primes.*

Question 5

Let $f(n) = (-1)^n$. Find all f-perfect numbers n. Study the general case that $f(n) = \omega^n$, where $\omega \in \mathbb{C}$ is a root of unity.

Solution. In this case we will consider two functions $(-1)^n$ and i^n.

Theorem 5.1. *All f -perfect numbers are 8, p and $4p$ where p is an odd prime for $f(n) = (-1)^n$.*

Proof. Let $n = \prod_{p_i \in \mathbb{P}} p_i^{\alpha_i}$, β is the maximal power of 2 such that $2^\beta | n$. Let us calculate $\sum_{d|n} f(d)$:

$$\sum_{d|n} (-1)^d = \sum_{d|n,\, 2|d} 1 - \sum_{d|n,\, 2\nmid d} 1 = \beta \prod_{p_i \neq 2} (\alpha_i + 1) - \prod_{p_i \neq 2} (\alpha_i + 1) = (\beta - 1) \prod_{p_i \neq 2} (\alpha_i + 1).$$

Since $2f(n) = \pm 2$, so $0 \leq \beta \leq 3$ and $\beta \neq 1$.

1) If $\beta = 3$ then $\prod_{p_i \neq 2} (\alpha_i + 1) = 1$, i.e. $n = 8$.

2) If $\beta = 2$ then $\prod_{p_i \neq 2} (\alpha_i + 1) = 2$, i.e. $n = 4p$ where p is an odd prime.

3) If $\beta = 0$ then $-1 \prod_{p_i \neq 2} (\alpha_i + 1) = -2$, i.e. $n = p$ where p is an odd prime. Theorem 5.1 is proved.

Theorem 5.2. *Let $f(n) = i^n$ and $(4k + 3), (4t + 1) \in \mathbb{P}$. All f-perfect numbers are $2(4k + 3)$, $8(4k + 3)$ and $(4t + 1)n^2$, where all prime divisors of n are in the form of $4k + 3$.*

Proof. Let $f(n) = i^n$. Then

1) $f(n) = i, n \equiv 1 (\mod 4)$;

2) $f(n) = -1, n \equiv 2 (\mod 4)$;

3) $f(n) = -i, n \equiv 3 (\mod 4)$;

4) $f(n) = 1, n \equiv 0 (\mod 4)$.

Consider the following cases:

I. $n \equiv 0(\mod 4)$. $2f(n) = 2$, i.e. the number of divisors that are equivalent 1 modulo 4 is equal the number of divisors that are equivalent 3 modulo 4. Let a and b ne the numbers of divisors that are equivalent 0 and 2 modulo 4 respectively then $a - 2 = b$. Let us show that n is $8(4k + 3)$ where $(4k + 3) \in \mathbb{P}$. Let d be the number of odd divisors of n (from below it follows that $d \geq 2$) and k be a maximal power of 2 dividing n. Now $a = (k - 1)d$ and $b = d$, $2 = (k - 2)d$, there is only one possible solution: $d = 2$, $k = 3$.The second odd divisor is $(4k+3) \in \mathbb{P}$.

II. $n \equiv 2(\mod 4)$. $2f(n) = -2$, i.e. the number of divisors that are equivalent 1 modulo 4 is equal the number of divisors that are equivalent 3 modulo 4. Let a and b ne the numbers of divisors that are equivalent 2 and 0 modulo 4 respectively then $a - 2 = b$. Let us show that n is $2(4k + 3)$ where $(4k + 3) \in \mathbb{P}$. Let d be the number of odd divisors of n (from below it follows that $d \geq 2$) and k be a maximal power of 2 dividing n. Now $a = d$ and $b = (k - 1)d$, $-2 = (k - 2)d$, there is only one possible solution: $d = 2$, $k = 1$. The second odd divisor is $(4k + 3) \in \mathbb{P}$.

III. n is odd. Let $n = \prod\limits_{p_i \in \mathbb{P}} p_i^{\alpha_i}$. Let $\delta(n)$ be a difference between the number of $4k + 1$ divisors of n and the number of $4k + 3$ divisors of n. By analogy with previous cases we need to describe all numbers with $\delta(n) = \pm 2$. Let $L(n)$ be the number of $4k + 1$ divisors of n, $l(n)$ be the number of $4k + 1$ divisors of n such that all their prime divisors are $4t + 1$, $R(n)$ be the number of $4k + 3$ divisors of n, $r_3(n)$ be the number of $4k + 3$ divisors of n such that all their prime divisors are $4t + 3$, $r_1(n)$ be the number of $4k + 1$ divisors of n such that all their prime divisors are $4t + 3$ (1 is included).

Note that $L(n) = l(n)r_1(n)$ and $R(n) = l(n)r_3(n)$. Now $|\delta(n)| = l(n)|r_1(n) - r_3(n)|$.

So we need to estimate $r_1(n) - r_3(n)$.

Proposition 5.1. *If all prime divisors of n be of the form $(4k + 3)$ then $\delta(n) = 1$ if n is a perfect square and 0 otherwise.*

Proof. Let $n = \prod\limits_{p_i \in \mathbb{P}} p_i^{\alpha_i}$. Let $d = \prod\limits_{p_i \in \mathbb{P}} p_i^{\beta_i}$ be a divisor of n . Note that if $\sum\limits_i \beta_i$ is even then $d \equiv 1(\mod 4)$ and if $\sum\limits_i \beta_i$ is odd then $d \equiv 3(\mod 4)$.

Lemma 5.1 *Let $n = \prod\limits_{p_i \in \mathbb{P}} p_i^{\alpha_i}$ and all prime divisors of n are of the form $4k+3$. If at least one α_i is odd then $\delta(n) = 0$.*

Proof. Suppose that α_j is odd. We need to show that the number of $4k + 1$ divisors is equal to the number of $4k + 3$ divisors. To do it we divide all divisors into groups such that a and b are in the same group iff $a/b = p_j^k$ $k \in \mathbb{Z}$. Consider such group. Let g be gcd of all numbers in this group. Note that one of the numbers gp_j^q and gp_j^{q+1} are of the form $4k + 1$ and another of the form $4k + 3$.

Since α_j is odd, the set $\{1, p_j, p_j^2, \dots\}$ contains even number elements. So our group contains equal numbers of $4k+1$ and $4k+3$ divisors. So every group contains equal numbers of $4k+1$ and $4k+3$ divisors. Now $\delta(n) = 0$. Lemma 5.1 is proved.

From lemma 5.1 it follows that if n is not a perfect square then $\delta(n) = 0$.

Lemma 5.2 *Let all prime divisors of a and b be of the form $(4k+3)$. If $(a;b) = 1$, $\delta(a) = \delta(b) = 1$ then $\delta(ab) = 1$.*

Proof. We have $\delta(a) = \delta(b) = r_1(a) - r_3(a) = r_1(b) - r_3(b) = 1$. It easy to see that $\delta(ab) = r_1(a)r_1(b) + r_3(a)r_3(b) - r_1(a)r_3(b) - r_3(a)r_1(b) = r_1(a)(r_1(b) - r_3(b)) + r_3(a)(r_3(b) - r_1(b)) = r_1(a) - r_3(a) = 1$. Lemma 5.2 is proved.

Let n be a perfect square. Note that $\delta(p^{2i}) = 1, i \in \mathbb{Z}_+, p = (4k+3), p \in \mathbb{P}$. This and n is perfect square imply $\delta(n) = 1$. Proposition 5.1 is proved

Let n be f-perfect. From proposition 5.1 and $\delta(n) \neq 0$ it follows that $r_1(n) - r_3(n) = 1$. Now $l(m) = 2$. So m has only one prime $4k+1$ divisor p and the power of it is 1, m/p is a perfect square and all prime divisors of m/p are in the form of $4k+3$. Theorem 5.2 is proved.

Question 6

Let $f(n) = C_{2012}^n$. Find all f-perfect numbers n. Study the general case where 2012 is replaced by a natural number m.

Solution. We will show that for infinitely many n there is C_n-perfect number.

Theorem 6.1 Let $f(n) = C_m^n$. There is f-perfect numbers for $m = p+1$, where $p \in \mathbb{P}$.

Proof. $C_{p+1}^p = C_{p+1}^1 = \sum\limits_{d|p, d<p} C_{p+1}^d$. Theorem 6.1 is proved.

Remark 6.1 *There is only one C_{2012}-perfect number - 2011.*

Proof. Using Wolfram Mathematica and the following commands we obtain it.

$$Div[n\_] := \sum_{i=1}^{n} (Boole[Mod[n,i] == 0] * Binomial[2012, i])$$

$$Dqwer[n\_] := Div[n] - 2Binomial[2012, n]$$

$$Table[\{i, Dqwer[i]\}, \{i, 1, 2012\}]//TableForm$$

Question 7

Consider other arithmetic functions f and find sufficient and/or necessary conditions for a number to be f-perfect.

Solution. We considered such functions in questions 2 and 4.

Question 8

Recall that a pair (m, n) of positive integers is said to be *amicable* if $n = \sum_{d|m, 1 \le d \le m-1} d$ and $m = \sum_{d|n, 1 \le d \le n-1} d$. An example is (220, 284). For an arithmetic function f, give a reasonable definition for a pair (m, n) of positive integers to be f-amicable. For various arithmetic functions f, find f-amicable pairs or prove that no f-amicable pair exists.

Solution. We introduced the following definition of f-amicable pairs.

Definition. Let $f(n)$ be an arithmetic function. We shall call the pair (a, b) of natural numbers the f-amicable if $a \neq b$ and

$$f(b) + f(a) = \sum_{d|b} f(d) = \sum_{d|a} f(d).$$

Theorem 8.2. *There is no J_a-amicable numbers.*

Proof. Since $J_a(n) + J_a(m) = \sum_{d|n} J_a(d) = n^a = \sum_{d|m} J_a(m) = m^a$, $m = n$, a contradiction.

Theorem 8.5. *Let $f(n) = (-1)^n$. All f-amicable pairs are $(8; 4p)$, $(4p; 4q)$ and $(p; q)$ where $p \neq q$ are primes.*

Proof. We have four cases:

1) $f(b) + f(a) = 0$, i.e. $\sum_{d|b} f(d) = \sum_{d|a} f(d) = 0$ by analogy with the proof of theorem 5.1 we have that a and b are divisible by 2 and not by 4. But now $f(b) + f(a) = 2$, a contradiction.

2) $f(b) + f(a) = 2$, ie $\sum_{d|b} f(d) = \sum_{d|a} f(d) = 2$. This case was considered in the proof of theorem 5.1. We have two different solutions 8 and $4p$, $p \in \mathbb{P}$. So we have that f-amicable pairs are $(8; 4p)$, $(4p; 4q)$ where $p \neq q$ are primes.

3) $f(b) + f(a) = -2$, i.e. $\sum_{d|b} f(d) = \sum_{d|a} f(d) = -2$. This case was considered in the proof of theorem 5.1. We have one solution p, $p \in \mathbb{P}$. So we have that f-amicable pairs are $(p; q)$ where $p \neq q$ are primes.

4) Since $f(n) \neq 0$ $\forall n \in \mathbb{N}$, $f(b) + f(a) \neq \pm 1$.

Generalizations

Let A and B be the sets closed under two associative commutative operations: additive $(+)$ and multiplicative $(\cdot)$, $A \subseteq B$ and there is $e \in A$ such that $ea = a, \forall a \in A$.

Let $a, b \in A$. We shall call that a is divisible by b if there is $c \in A$ such that $bc = a$.

Let f be mapping from A to B.

We say that an element $n \in A$ is *f-perfect* if $f(n) + f(n) = \sum\limits_{d|n} f(d)$.

An element $n \in A$ is called perfect if n is f-perfect for $A = B$ and $f(n) = n$.

Theorem 9.1. *There is only one perfect number in Z_n and it is 0 if n is odd and there is no perfect numbers in Z_n if n is even.*

Proof. Let $a \in \mathbb{N}$ and $\gcd(a, n)$=d. Note that all divisors of $\bar{a} \in Z_n$ are numbers $\bar{a}_i$ with $\gcd(a_i, n) | \gcd(a, n)$. But if $\gcd(a_i, n) | \gcd(a, n)$ then $\gcd(n - a_i, n) | \gcd(a, n)$. It means that if $\bar{a}_i | \bar{a}$ then $-\bar{a}_i | \bar{a}$

I. n is odd. It means that $-\bar{a} \neq \bar{a}$ for all $\bar{a} \in Z_n$ and $\bar{a} \neq 0$. Now $\sum\limits_{\bar{d}|\bar{a}} \bar{d} = 0 = 2\bar{a}$. It means that $\bar{a} = 0$.

II. n is even. If $\bar{a}$ is not divisible by $\overline{\frac{n}{2}}$ then $-\bar{a}_i \neq \bar{a}_i$ for every divisor $\bar{a}_i$ of $\bar{a}$. By analogy $\sum\limits_{\bar{d}|\bar{a}} \bar{d} = 0 = 2\bar{a}$, a contradiction. If $\bar{a}$ is divisible by $\overline{\frac{n}{2}}$ then $\sum\limits_{\bar{d}|\bar{a}} \bar{d} = 2\bar{a} = 0 = \overline{\frac{n}{2}}$, finally we get a contradiction. The theorem is proved.

Finally we construct the set with two commutative operations: additive $(+)$ and multiplicative $(\cdot)$ such that every number of it is perfect.

Example 9.1. *Let A be the set such that it is additive abelian group A_a with neutral element e_a and A is multiplicative abelian group A_m with neutral element $e_m = e_a$. A_a and A_m both are generated by involutions and $\sum\limits_{x \in A_a} = e_a$ and $\prod\limits_{x \in A_m} = e_m$.*

In particular let $A = \{e, a, b, c\}$, $a + b + c = abc = a^2 = b^2 = c^2 = a + a = b + b = c + c = e$,$ea = a + e = a$,$eb = e + b = b$,$ce = c + e = c$.

Let $a \in A$. Note that all elements of A are divisors of a. Since sum of all elements of A and $a + a$ are equal e_a, so a is perfect.

References

[1] *Sivaramakrihnan R.*, Classical theory of arithmetic functions, Marcel Dekker, 1989, 406 p.