Problem 9: A Topological Problem
Team: Russia

Abstract

1)It was shown that for the operations int, cl we can obtain 7 distinct sets. Also this result is true for any finite dimensional spase($\mathbb{R}^n$).

2)one of the theorems shown that equality from the question 2 holds. And both cases offered in the condition of the second question has been studied:

1) in the case $int(conv(A)) = \emptyset$, It was proved that for finite dimensional space by using all operations we can get 6 distinct sets, and if A boarded then we cn get only five sets and for infinite dimensional space it was proved that we can get at most fourteen distinct sets

2)in the case $int(conv(A)) \neq \emptyset$, it was proved that we can get seventeen distinct sets and if A is boarded we can get forteen sets

THEOREM 1. If B is open, then $cl(B) = cl(int(cl(B)))$.
Proof. we proved this equality by two inclucions:

$$int(cl(B)) \subseteq cl(B) \Rightarrow cl(int(cl(B))) \subseteq cl(cl(B)) = cl(B).$$

$$cl(B) \supseteq B \Rightarrow int(cl(B)) \supseteq int(B) = B \Rightarrow$$
$$\Rightarrow cl(int(cl(B))) \supseteq cl(B).$$

CONSEQUENCE. For any A

$$cl(int(A)) = cl(int(cl(int(A)))).$$

THEOREM 2. If B is closed, then $int(B) = int(cl(int(B)))$.
Proof.

$$(a) \quad cl(int(B)) \supseteq int(B) \Rightarrow$$
$$\Rightarrow int(cl(int(B))) \supseteq int(int(B)) = int(B).$$
$$(b) \quad int(B) \subseteq B \Rightarrow cl(int(B)) \subseteq cl(B) = B \Rightarrow$$
$$\Rightarrow int(cl(int(B))) \subseteq int(B).$$

QONCEQUENCE For any A

$$int(cl(A)) = int(cl(int(cl(A)))).$$

Now let's study properties of the *conv* operation.

THEOREM 3. For any A

$$cl(conv(cl(A))) = cl(conv(A)).$$

Proof. First inclusion If

$$conv(A) \subseteq conv(cl(A)),$$

then

$$cl(conv(A)) \subseteq cl(conv(cl(A))).$$

Then second inclusion: $conv(cl(A))) \subseteq cl(conv(A))$.

Let $x \in conv(cl(A))$. then $x = \sum\limits_{i=1}^{n} t_i x_i$ for some $x_1,\ x_2,\ \ldots,\ x_n \in cl(A)$ and $t_1,\ t_2,\ \ldots,\ t_n \in (0,1)$ such that $t_1 + t_2 + \ldots + t_n = 1$. Due to $x \in cl(conv(A))$, for any neighbourhood W of zero

$$(x + W) \cap conv(A) \neq \emptyset.$$

let's condider neughbour fo zero W.Then exist balanced neighbourhood of zero U, such that

$$\sum_{i=1}^{n} U \subseteq W.$$

$x_i \in cl(A)$, thus

$$(x_i + U) \cap A \neq \emptyset.$$

let $y_i \in (x_i + U) \cap A$, and $y = \sum_{i=1}^{n} t_i y_i$. $y_i \in A$ then $y \in conv(A)$. But

$$y = \sum_{i=1}^{n} t_i y_i \in \sum_{i=1}^{n} ((x_i + U) \cap A) t_i \subseteq \sum_{i=1}^{n} (x_i + U) t_i =$$

$$= \sum_{i=1}^{n} x_i t_i + \sum_{i=1}^{n} U t_i = x + \sum_{i=1}^{n} U t_i \subseteq x + W$$

due to balancity of U and $t_i < 1$. Thus $y \in x + W$. But $y \in conv(A)$, so

$$conv(A) \cap (x + W) \neq \emptyset$$

for any W. So $x \in cl(conv(A))$ or

$$conv(cl(A)) \subseteq cl(conv(A)$$

. It means that

$$cl(conv(cl(A))) \subseteq cl(cl(conv(A))) = cl(conv(A)).$$

THEOREM 4. If B convex set with not empty, interior and U is open and $B \cap U \neq \emptyset$, then exist open W such that $W \subseteq U \cap B$.

Proof. At first notice that if $x, y \in B$ and $s \in [0, 1]$, then

$$s(y - x) + x = sy + (1 - s)x \in B,$$

because B convex. Now let $x \in B \cap U$, and $y \in int(B)$. Then $y - x$ is limited $\Rightarrow$ exist $t > 1$ such that $y - x \in t(U - x)$ ($U - x$ — neighbourhood of zero), that is $\frac{1}{t}(y - x) \in U - x$ or $\frac{1}{t}(y - x) + x \in U$. $\frac{1}{t} < 1 \Rightarrow \frac{1}{t}(y - x) + x \in B$, or (and) $\frac{1}{t}(int(B) - x) + x \subseteq B$. let $W =$

$U \cap (\frac{1}{t}(int(B) - x) + x)$ — not empty (because $\frac{1}{t}(y - x) + x \in W$) open set, with $W \subseteq U \cap B$. So W is the set which wee had to wind

DEFENITION. Let B be internally dense in V, if $int(B)$ dence в V,or in another words for any open U such that $U \cap V$ isn't empty $int(U \cap B)$ isn't empty too.

THEOREM 5. If B — convex set with notempty interior, then B internally dense in $cl(B)$.

Proof. Let $U \subseteq cl(B)$ and $x \in U \subseteq cl(B)$. Then $U \cap B \neq \emptyset$. Using theorem 4, exist open $W \subseteq U \cap B$, but $W \subseteq int(B)$, that is $W \subseteq U \cap int(B)$ — what we had to prove

THEOREM 6. If B is convex and internally dense in open U, then $U \subseteq B$.

Proof. Let theorem isn't correct. So $x \in U$ and $x \notin B$. Then exidt balanced neighbourhood of zero W such that$x + W \subseteq U$. Then, for any $y \in W$ such that $x + y \in B$, $x - y \notin B$ is true (in other case using convexity if B $\frac{1}{2}(x + y) + \frac{1}{2}(x - y) = x \in B$?!?). B is internally dense in $U \Rightarrow$ exists $V \subseteq W$ such that $x + V \subseteq B$, but then $(x - V) \cap B = \emptyset$, that is a contradiction, because B is internally dense . So for any $x \in U \Rightarrow x \in B$.

THEOREM 7. If B — convex set with not empty interior, then $int(cl(B)) = int(B)$.

Proof. (a) $B \subseteq cl(B) \Rightarrow int(B) \subseteq int(cl(B))$

(b) Let $x \in U \subseteq int(cl(B)) \Rightarrow U \subseteq cl(B)$. Using consequence 5, B is internally dense in $cl(B) \Rightarrow B$ is internally dense in U,and using theorem 6 $U \subseteq B \Rightarrow U \subseteq int(B) \Rightarrow x \in int(B)$ — xnd.

THEOREM 8. If B — convex set with not empty interior, then $cl(int(B)) = cl(B)$.

Proof.

(a) $int(B) \subseteq B \Rightarrow cl(int(B) \subseteq cl(B)$.

(b) Let $x \in cl(B)$. Consider neighboorhood of U. $(x + U) \cap B \neq \emptyset \Rightarrow$ then by theorem 4,exists open set $W \subseteq (x + U) \cap B$. Thus $W \subseteq int(B) \Rightarrow (x + U) \cap int(B) \neq \emptyset$ for any neighbourhood of zero U $\Rightarrow x \in cl(int(B)$.

CONSEQUENCE 9. If $int(conv(A)) \neq \emptyset$, then

$$int(conv(A)) = int(conv(cl(A))).$$

By theorem 3, $cl(conv(A)) = cl(conv(cl(A)))$. Using theorem 7, we achieve that

$$int(conv(A)) = int(cl(conv(A))) =$$

$$= int(cl(conv(cl(A)))) = int(conv(cl(A))).$$

Now using all previous theorem we can study first question of the problem 9: How many distinct sets can be obtained using operations *int* and *cl*?

It is obvious that $cl(cl(A)) = cl(A)$ and $int(int(A)) = int(A)$. So it is unnessesary to apply any of this operations in a row thus we should to apply operations *int* and *cl* in turn. By the theorems 1,2 it is unnessesary to apply more than three different operations in a row. So maximum numberof sets we can obtain is 7:

$$A, cl(A), int(A), int(cl(A)), cl(int(A)),$$

$$cl(int(cl(A))), int(cl(int(A)))$$

And for exmple if $A = \{0\} \cup [1,2) \cup (2,3) \cup ((4,5) \cap \mathbb{Q})$ and $X = \mathbb{R}$, then all obtained sets are distinct. So answer on the question 1 is 7

Now solve question 2 and 3 when we also can apply $conv(A)$.

of course of A is convex then $int(A)$ and $cl(A)$ are convex too. So we should use *conv* one time in a row. At this point let's consider two cases as it was offered in the problem: $int(conv(A)) = \emptyset$ and $int(conv(A)) \neq \emptyset$.

$int(conv(A)) = \emptyset$. So $int(A) = \emptyset$, thus first operation should be *cl*, and then *int*. So can apply operation *conv* (if it exists) only to following four sets: $A, cl(A), int(cl(A)), cl(int(cl(A)))$. let find out what we can apply after operation *conv*:

1) A–

$$conv(A), cl(conv(A)), int(cl(conv(A))), cl(int(cl(conv(A)))),$$

because $int(conv(A)) = \emptyset$.

2)$cl(A)$ –

$$conv(cl(A)), int(conv(cl(A))),$$

because $cl(conv(cl(A))) = cl(conv(A))$, and if $int(conv(cl(A))) \neq \emptyset$, then $cl(int(conv(cl(A)))) = cl(conv(cl(A))) = cl(conv(A))$.

3)Similarly for $int(cl(A))$ –

$$conv(int(cl(A))), cl(conv(int(cl(A)))),$$

4)and for $cl(int(cl(A)))$ –

$$conv(cl(int(cl(A)))), int(conv(cl(int(cl(A))))).$$

Thus if $int(conv(A)) = \emptyset$, we can achieve only following 14 sets:

$$\emptyset, A, conv(A), cl(conv(A)), int(cl(conv(A))),$$

$$cl(A), conv(cl(A)), int(conv(cl(A))),$$

$$int(cl(A)), conv(int(cl(A))), cl(conv(int(cl(A)))),$$

$$cl(int(cl(A))), conv(cl(int(cl(A)))), int(conv(cl(int(cl(A))))).$$

Due to $A \subseteq \mathbb{R}^n$ in the condition of the problem, if $int(conv(A)) = \emptyset$, then if we apply operation *int* anywhere then we achieve empty set

Proof: let's take n and proof that exists $X - (n-1) - dimensional\ subspace\ of\ R^n\ conv(A) \subseteq X$. By the theorem from the book [1]

$$conv(A) = \bigcup_{k \leq n} \bigcup_{x1,...x_k \in A} \Delta_k(x_1, ..., x_k),\ where\ \Delta_k(x_1, ..., x_{k+1})k - dimensional\ simplex$$

$int(conv(A)) = \emptyset$ and only if $k = n\ \ then\ int(Delta_k) \neq \emptyset$. So there is no n-dimensional simlexes by the points from A. Thus for any $n+1$ points from A exists $(n-1)$-dimensional subspace of $\mathbb{R}^n$. So exist $(n-1)$ dimensional subspace X,such that $conv(A) \subseteq X$. It's easy no notice than if $B \subseteq Y$ then $conv(B), cl(B) \subseteq Y$ if Y is space. So if $A \subseteq X$ then $conv(A), cl(A)$ are subsets of X, because subspace is space. So their interiors will be empty sets. Thus if we consider finitely dimensional spaces then by operations $int, conv, cl$ we can obtain six sets :

$$\emptyset, A, conv(A), cl(conv(A))$$

$$cl(A), conv(cl(A)),$$

$$A = \{(x, 1)|x \in (-\infty, 0) \cup 1 \cup [2, 3] \cup (4, 5) \cup ((5, 6) \cap \mathbb{Q}) \cup (7, 8)\}$$

As you see we can get this six sets But if A is limited then we can obtain only five sets because $cl(conv(A)) = conv(cl(A))$

Proof. (two inclusions)

1)$conv(cl(A)) \subseteq cl(conv(A))$ was proved in theorem 3;

2)$cl(conv(A)) \subseteq conv(cl(A))$:$conv(A) \subseteq conv(cl(A))$, $cl(A)$-compact set then $conv(cl(A))$ is compact set too, so it is closed. Thus $cl(conv(A)) \subseteq conv(cl(A))$

Now let's consider more interesting case: if

$$int(conv(A)) \neq \emptyset.$$

Before operation *conv* we can apply any combination of the operations *cl* and *int*. So let's study what we can apply after the operation *conv* to get distinct sets:

1) A –

$$conv(A), cl(conv(A)), int(conv(A)),$$

because by theorems 7,8

$$int(cl(conv(A))) = int(conv(A))$$

and

$$cl(int(conv(A))) = cl(conv(A)).$$

2)$int(A)$ –

$$conv(int(A)), cl(conv(int(A))).$$

3) $int(cl(A))$ –

$$conv(int(cl(A))), cl(conv(int(cl(A)))).$$

4) $cl(A)$ –

$$conv(cl(A)),$$

because $cl(conv(cl(A))) = cl(conv(A))$,and by the consequence 9

$$int(conv(cl(A))) = int(conv(A)).$$

5)similarly for $cl(int(A))$ and $cl(int(cl(A)))$ we get

$$conv(cl(int(A))) \text{ and } conv(cl(int(cl(A)))) \text{ respectively .}$$

6) $int(cl(int(A)))$ we get

$$conv(int(cl(int(A)))) = int(conv(cl(int(cl(int(A)))))) =$$

$$= int(conv(cl(int(A)))) = int(conv(A)).$$

thus if $int(conv(A)) \neq \emptyset$ we can achieve only these distinct sets:

$$A, conv(A), cl(conv(A)), int(conv(A)),$$

$$int(A), conv(int(A)), cl(conv(int(A))),$$

$$cl(A), conv(cl(A)),$$

$$cl(int(A)), conv(cl(int(A))),$$

$$int(cl(A)), conv(int(cl(A))), cl(conv(int(cl(A)))),$$
$$cl(int(cl(A))), conv(cl(int(cl(A)))),$$
$$int(cl(int(A))).$$

Now make an example if $A \subset R^2$, when all this seventeen sets are distinct

$$A_1 = \{(x,\ y) \mid x \in (-\infty,\ 0),\ y \in (0,\ 1]\ \},\quad A_2 = \{(x,\ y) \mid x \in [1,\ 2),\ y \in (0,\ 1]\ \},$$
$$A_3 = \{(x,\ y) \mid x \in (2,\ 3),\ y \in (0,\ 1]\ \},\quad A_4 = \{(x,\ y) \mid x \in (4,\ +\infty) \cap Q,\ y \in (0,\ 1] \cap Q\ \},$$
$$A_5 = \{(x,\ y) \mid (x-2)^2 + (y-3)^2 \leq 1\ \},\quad A_6 = \{(3,\ 5)\}.$$

let's take $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$. than

$$conv(A) = \{(x,\ y) \mid x \in R,\ y \in (0,\ 5)\ \} \cup \{(3,\ 5)\},$$
$$cl(conv(A)) = \{(x,\ y) \mid x \in R,\ y \in [0,\ 5]\ \},$$
$$int(conv(A)) = \{(x,\ y) \mid x \in R,\ y \in (0,\ 5)\ \},$$
$$int(A) = \{(x,\ y) \mid x \in (-\infty,\ 0),\ y \in (0,\ 1)\ \} \cup \{(x,\ y) \mid x \in (1,\ 2),\ y \in (0,\ 1)\ \} \cup$$
$$\cup \{(x,\ y) \mid x \in (2,\ 3),\ y \in (0,\ 1)\ \} \cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 < 1\ \},$$
$$conv(int(A)) = \{(x,\ y) \mid x \in (-\infty,\ 2), y \in (0,\ 4)\ \} \cup \{(x,\ y) \mid x \in (-\infty,\ 3),\ y \in (0,\ 3)\ \} \cup$$
$$\cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 < 1\ \},$$
$$cl(conv(int(A))) = \{(x,\ y) \mid x \in (-\infty,\ 2], y \in [0,\ 4]\ \} \cup \{(x,\ y) \mid x \in (-\infty,\ 3],\ y \in [0,\ 3]\ \} \cup$$
$$\cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 \leq 1\ \},$$
$$cl(A) = \{(x,\ y) \mid x \in (-\infty,\ 1],\ y \in [0,\ 1]\ \} \cup \{(x,\ y) \mid x \in [1,\ 3],\ y \in [0,\ 1]\ \} \cup$$
$$\cup \{(x,\ y) \mid x \in [4,\ +\infty),\ y \in [0,\ 1]\ \} \cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 \leq 1\ \} \cup \{(3,\ 5)\},$$
$$conv(cl(A)) = \{(x,\ y) \mid x \in R,\ y \in [0,\ 5)\ \} \cup \{(3,\ 5)\},$$
$$cl(inf(A)) = \{(x,\ y) \mid x \in (-\infty,\ 0],\ y \in [0,\ 1]\ \} \cup \{(x,\ y) \mid x \in [1,\ 3],\ y \in [0,\ 1]\ \} \cup$$
$$\cup \{(x,\ y) \mid (x - 2^2 + (y-3)^2 \leq 1\ \},$$
$$conv(cl(int(A))) = \{(x,\ y) \mid x \in (-\infty,\ 2],\ y \in [0,\ 4)\ \} \cup \{(x,\ y) \mid x \in (-\infty,\ 3],\ y \in [0,\ 3]\ \} \cup$$
$$\cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 \leq 1\ \},$$
$$int(cl(A)) = \{(x,\ y) \mid x \in (-\infty,\ 0),\ y \in (0,\ 1)\ \} \cup \{(x,\ y) \mid x \in (1,\ 3),\ y \in (0,\ 1)\ \} \cup$$
$$\cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 < 1\ \} \cup \{(x,\ y) \mid x \in (4,\ +\infty),\ y \in (0,\ 1)\ \},$$
$$conv(int(cl(A))) = \{(x,\ y) \mid x \in R,\ y \in (0,\ 4)\ \},$$
$$cl(conv(int(cl(A)))) = \{(x,\ y) \mid x \in R,\ y \in [0,\ 4]\ \},$$
$$cl(int(cl(A))) = \{(x,\ y) \mid x \in (-\infty,\ 0],\ y \in [0,\ 1]\ \} \cup \{(x,\ y) \mid x \in [1,\ 3],\ y \in [0,\ 1]\ \} \cup$$
$$\cup \{(x,\ y) \mid x \in [4,\ +\infty),\ y \in [0,\ 1]\ \} \cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 \leq 1\ \},$$
$$conv(cl(int(cl(A)))) = \{(x,\ y) \mid x \in R,\ y \in [0,\ 4)\ \} \cup \{(2,\ 4)\},$$
$$int(cl(int(A))) = \{(x,\ y) \mid x \in (-\infty,\ 0),\ y \in (0,\ 1)\ \} \cup \{(x,\ y) \mid x \in (1,\ 3),\ y \in (0,\ 1)\ \} \cup$$
$$\cup \{(x,\ y) \mid (x-2)^2 + (y-3)^2 < 1\ \}.$$

You can sure that all this sets are distinct.

It is obvious that for n=1 we cann't formulated similar example.

If A is boardered set then as we had already proved $cl(conv(A)) = conv(cl(A))$. So we can achieve only these 14 distinct sets:

$$A, conv(A), int(conv(A)),$$

$$int(A), conv(int(A)),$$
$$cl(A), conv(cl(A)),$$
$$cl(int(A)), conv(cl(int(A))),$$
$$int(cl(A)), conv(int(cl(A))),,$$
$$cl(int(cl(A))), conv(cl(int(cl(A)))),$$
$$int(cl(int(A))).$$

Using literature:

[1] Convex sets, Leijhtvejs, Nauka, M, 1985 (Leichtweiss, Konvexe Mergen, Springer, Berlin, 1980)