PROBLEM 8: Point On Curves

Team: Russia

Abstract

Point 1.In this paper we formulated and proved theorem about the monotony-increasing and monotony-decreasing functions. Also we formulated theorems for part-linear and part-monotony functions. So now the problem, which were stated in the point 1 is now solved for these types of functions.

Point 2. We formulated theorem for the case of $C \geq 2\sqrt{2}$ and $\alpha \geq 1$.

Chapter 1.

Part 1. Monotonous increasing functions.

Definition 1. $f:[0,1] \to \mathbb{R}$ –continuous function. The partition of the area of the domain we call "good" of order n, if $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ and the expression $|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$ is not depend on i.

Theorem 1. If $f:[0,1] \to \mathbb{R}$ –continuos monotony increasing function, then there is a good partition n for all natural n.

Proof. We'll build the partition of order n for this function. By the condition, $t_0 = 0$. We can also think that $f(t_0) = f(0) = 0$. (If the function $f(0) \neq 0$ and $f(0) + l = 0$, then we'll consider the function g such that $g(0) = f(0) + l$.)

Let fix $C = \frac{f(1)+1}{n}$. Let us consider the expression $|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$ when $i = 1$:
$|f(t_1) - f(t_0)| + |t_1 - t_0| = |f(t_1)| + |t_1| = (using\ the\ monotony) = f(t_1) + t_1$

Make equal $f(t_1) + t_1$ to C. From this expression we can define t_1 synonymous, because the function $f(t_i) + t_i \to \infty$ when $t_i \to \infty$ and it injective.
Then let us consider the expression $|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$ when $i = 2$. It also should be equal to C.

$$|f(t_2) - f(t_1)| + |t_2 - t_1| = f(t_2) - f(t_1) + t_2 - t_1 = \mathrm{C}$$

Using $f(t_1) + t_1 = C$, we get: $f(t_2) + t_2 = 2\mathrm{C}$. Analogous we can build the partition for the great i. It is obvious that $f(t_n) + t_n = n\mathrm{C}$. Using the bijectivity and continous of this function we can always make such partition. It is clear that $t_n = 1$, by the choosing of C. So for any positive n we can always find the partition of the order n.

Part 2. Chains and other a lot.

Theorem 2. If $f:[0,1] \to \mathbb{R}$ –continuos part-linear function, there is a good partition n for all natural n.

Proof. Let our chain is defined by the set of functions $f_1, f_2, \ldots, f_s$, where f_i is defined on the section(stretch) I_i. As in the Theorem 1 we can thinm that $f_1(0) = 0$ (If it is not true, we can move all the function up or down to achieve this condition, without the loss of the contionuos.) Then we can think that $f_1(x) = a_1 x$.

Let us consider the case when $a_1 \geq 0$. Fix $C = \frac{f_s(1)+1}{n}$. Consider the expression $|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$ when $i = 1$:

$$|f_1(t_1) - f_1(t_0)| + |t_1 - t_0| = a_1 t_1 - a_1 t_0 + t_1 - t_0 = a_1 t_1 + t_1 = C$$

Then $t_1 = \frac{C}{a_1+1}$. So we find t_1. Analogous we can find t_i, and $t_i = \frac{iC}{a_1+1}$. We've got to note that we consider only t_i, such that are in the domain area of this function. In the case when $a_1 \leq 0$ $t_i = \frac{iC}{1-a_1}$.
We will call the case when we consider the difference among the one function by "behaviour on the straingt line".

Then we'll consider the "joint" of the functions $f_1\ and\ f_2$. Let us suppose that $f_2(x) = a_2x + b_2$. And let us consider t_{i+1}:

$$|f_2(t_{i+1}) - f_1(t_i)| + |t_{i+1} - t_i| = |a_2t_{i+1} + b_2 - a_1t_i| + t_{i+1} - t_i = \mathrm{C}$$

It this expression we don't know only the t_{i+1}. We can find it by opening the module. Thus, by considering all the functions on the "behaviour on the straight line" and on the joints, we can find the partition of the order n for all natural n.

Theorem 3. If $f: [0,1] \to \mathbb{R}$ –continous part-monotony function with finite quantity of the interval of the monotony, then there is a good partition n for all natural n.

Proof. Let our funtion consist from the monotonious functions $f_1, f_2, \dots, f_s$, where f_i is defined on the interval I_i. As in the Theorem 1we can think that $f_1(0) = 0$ (If it is not true, we can move all the function up or down to achieve this condition, without the loss of the contionuos.) By the condition each function is monotonous onif area of the domain.

Fix $C = \frac{f_s(1)+1}{n}$. Consider the expression $|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$ when $i = 1$:

$$|f_1(t_1) - f_1(t_0)| + |t_1 - t_0| = |f_1(t_1)| + t_1 = C$$

Then we can define t_1 synonymous, using the monotony and injectivity on the area of the domain of this fumction. Analogous we can find t_i. We've got to note that we consider only such t_i, which are in the area of the domain of this function.

Then we consider the joint of the functions $f_1\ and\ f_2$. We'll consider t_{i+1}:

$$|f_2(t_{i+1}) - f_1(t_i)| + |t_{i+1} - t_i| = |f_2(t_{i+1}) - f_1(t_i)| + t_{i+1} - t_i = \mathrm{C}$$

In this expression we do not know only the t_{i+1}. We open the module. In the both cases we can find t_{i+1} synonymous, as f_2 is monotony. (Note. t_{i+1} can be out of the I_2. Then we'll consider f_3. And so on.) We've got to note the we can do the same actions for all functions, thus we can define the partition for all interval $[0,1]$. So for all natural n we can find the partition of the order n.

Consequence 1. Using Theorem 3 we can study points b) and c) of first part of problem.

Remark 1. There are not always good partition of order n for continuos part-monotony functions $f: [0,1] \to \mathbb{R}$ with infinite quantity of sections of monotony.

Chapter 2.

Lemma 1. For all natural i $\quad A_i \in \overline{B_1}(0)$.

Proof. $d(A_0, A_j) \le \left(\frac{j}{n}\right)^{\alpha} \le 1$. But $A_0 = (0,0)$, thus $d\big((0,0), A_j\big) \le 1$. So $A_i \in \overline{B_1}(0)$.

Теорема 4. If $C \ge 2\sqrt{2}$ for all natural n there is α -tight consequence, where $\alpha \ge 1$ and there exist C-fitting function.

Proof. Let us consider the function $F(A_i) = \frac{x_i{}^2 - y_i{}^2}{2}$. So

$$\left|F(A_j) - F(A_i) - x_j y_i + x_i y_j\right| = \left|\frac{x_j{}^2 - y_j{}^2}{2} - \frac{x_i{}^2 - y_i{}^2}{2} - x_j y_i + x_i y_j\right| =$$

$$= \left|\frac{x_j{}^2 - 2x_j y_i + y_i{}^2 - (x_i{}^2 - 2x_i y_j + y_j{}^2)}{2}\right| = \frac{\left|(x_j - y_i)^2 - (x_i - y_j)^2\right|}{2} =$$

$$= \frac{\left|(x_j - y_i + x_i - y_j)(x_j - y_i - x_i + y_j)\right|}{2} =$$

$$= \left|x_j - y_i + x_i - y_j\right| * \frac{\left|(x_j - x_i) + (y_j - y_i)\right|}{2} \le$$

$$\le \left|x_j - y_i + x_i - y_j\right| * \sqrt{\frac{(x_j - x_i)^2 + (y_j - y_i)^2}{2}} = \frac{\left|x_j - y_i + x_i - y_j\right|}{\sqrt{2}} * d(A_i, A_j) \le$$

$$\le using\ Lemma\ 1 \le \frac{4}{\sqrt{2}} * d(A_i, A_j) \le 2\sqrt{2}\left(\frac{j-i}{n}\right)^{\alpha}$$