PROBLEM 2: SEPARATING FUNCTIONS

Team:*Russia.*

ABSTRACT:

In this paper we have studied some facts about separating functions. In particular our research includes proofs of following statements:

- Part 1 of the problem was fully solved: we proved that for any tuple of positive integers exists f: such a number, that for any $t \geq f$ t can be obtained as linear combination of these integers with non-negative coefficients.
- Part 2 was also fully researched, the existence of separating function for any two positive integers x, y was proved.
- Part 3 was partially solved (subpart b for $GCD(a_1, a_2) = 2$)).

1. The existence of f for the case, when the tuple contains only two numbers, follows from part 2 of our research: at least one of such numbers exists because value of separating functions is defined and it is minimal among all of them. Let's assume that for any tuple containing n numbers exists f and prove that for any tuple of $n+1$ numbers $f^{'}$ exists. For $a_1, ..., a_n$ we'll define $b_i = \frac{a_i}{GCD(a_1,...,a_n)}$. Following definition, $GCD(b_1, ..., b_n) = 1$, so exists such a f that for any $b \geq f$ b can be obtained as a linear combination of these integers with non-negative coefficients, in particular we can take b coprime with a_{n+1}. $GCD(a_1, ..., a_n)$ also is coprime with a_{n+1} because in other case $a_1, ..., a_{n+1}$ aren't coprime, that contradicts the definition of the problem. So $b * GCD(a_1, ..., a_n)$ is coprime with a_{n+1} too. Following the case of two numbers, there is $f^{'}$, for that any number larger or equal than $f^{'}$ can be obtained as a linear combination of $b * GCD(a_1, ..., a_n)$ and a_{n+1} with non-negative coefficients. But, following definition, $GCD(a_1, ..., a_n) * b = GCD(a_1, ..., a_n) * (x_1 * b_1 + ... + x_n * b_n) = x_1 * a_1 + ... + x_n * a_n$ where x_i is non-negative, so, as any number q larger or equal than $f^{'}$ can be obtained as some $x * b + y * a_{n+1}$ where x, y are non-negative, q can also be obtained as $x * x_1 * a_1 + ... + x * x_n * a_n + y * a_{n+1}$, so $f^{'}$, the number we were searching for, exists.

2 To begin with, we prove lemma: if $a = r_1x + r_2y$ and $a = r_1^{'}x + r_2^{'}y$ where x, y - coprime integers, $r_i, r_i^{'}$ - integers, when exists such a d - integer, that $r_1^{'} = r_1 - dy$, $r_2^{'} = r_2 + dx$.
From $r_1x + r_2y = r_1^{'}x + r_2^{'}y$ follows that $(r_1 - r_1^{'})x = (r_2^{'} - r_2)y$. Left part must to divide by y, x is coprime with y so $r_1 - r_1^{'}$ must to divide by y that means that $r_1 - r_1^{'} = dy$ or just $r_1^{'} = r_1 - dy$. After that we rewrite first equation as $dxy = (r_2^{'} - r_2)y$, what means that $r_2^{'} - r_2 = dx$ or just $r_2^{'} = r_2 + dx$. So lemma is proved.

Theorem: separating function for two coprime numbers is $p(x, y) = xy - x - y - 1$. To begin with let's prove that for any a, b coprime integers $ab - a - b$ cannot be obtained as a linear combination of a and b with non-negative coefficients, so if for $f = ab - a - b + 1$ every $g \geq f$ can be obtained as a linear combination of a and b with non-negative coefficients, f will be minimal one. Without loss of generality we'll assume that $x < y$. $xy - x - y$ can be obtained as $y(x-1) - x$. Let's assume that it can also be assumed as $r_1x + r_2y$ where r_1, r_2 are non-negative. When, following lemma, exists such a d that $r_1 = (x-1) - dy, r_2 = dx - 1$. If d is positive, r_1 is negative, else r_2 is negative, that contradicts our assumption. Part of theorem has been proved. All what's left to prove is that $ab - a - b + k$ can be obtained as a linear combination of a and b with non-zero coefficients for any k.
At first, we'll consider the case of $k \leq xy$. Such an $r_1 \leq y$ exists that $r_1x \equiv kmody$. Following that, $r_2 = \frac{k - r_1x}{y}$ is integer, and $|r_2|$ is less than x. Than $k = r_1x - r_2y$ where r_1, r_2 are positive, and $xy - x - y + k$ can be obtained as $xy - x - y + r_1x - r_2y = y(x - r_2) + x(r_1 - 1)$ where, as shown before, both $x - r_2$ and $r_1 - 1$ are non-negative.
The case left is $k > xy$. When we'll define $d = kmodxy$, obtain $xy - x - y + d = r_1x + r_2y$ where r_1, r_2 are non-negative, as in previous case, and $xy - x - y + k$ will be obtained as $r_1x + (r_2 + \frac{k-d}{y})y$. So for any a, b $p(a, b)$ is the smallest of numbers f for what any of $q \geq f$ can be obtained from a, b as an linear combination with non-negative coefficients what means that $p(x, y)$ is separating function.

3 We'll consider case when $GCD(a_1, a_2) = 2$. In such a case $\frac{a_1}{2}$ is coprime with $\frac{a_2}{2}$, and according to part 2 there is $k = F(\frac{a_1}{2}, \frac{a_2}{2})$, minimal among numbers for which that any $q \geq k$ can be obtained as a linear combination of $\frac{a_1}{2}$ and $\frac{a_2}{2}$ with non-negative combination. When any even $p \geq 2k$ can be obtained from a_1, a_2 with the same coefficients, and $2k - 2$ cannot be obtained by such a way.
Any $m \geq 2k + F(2, a_3)$ can be obtained as a linear combination of a_1, a_2, a_3 with non-negative coefficients: let's prove it. $m \geq F(a_3, 2)$ then $m = r_1 2 + r_2 a_3$.
a) If $r_2 = 0$. Then m is even and greater than $2k$ so it can be obtained as $s_1 a_1 + s_2 a_2$, where $s_1 * \frac{a_1}{2} + s_2 * \frac{a_1}{2} = \frac{m}{2}$ - non negative coefficients, because $\frac{m}{2} > k = F(\frac{a_1}{2}, \frac{a_2}{2})$
b) If $r_2 = 1$. $F(2, a_3) = 2a_3 - a_3 - 2 + 1 = a_3 - 1$ according to part 2 of solution of problem. When $2r_1 + r_2 a_3 \geq 2k + a_3 - 1$ and $2r_1 \geq 2k - 1$ but r_1, k are integers, so $2r_1 \geq 2k$. When also m is even and greater than $2k$, so it can be obtained as in part a.
c) If $r_2 \geq 2$ when we'll change representation of m by following way: $m = (r_1 + s a_3) * 2 + (r_2 - 2s) * a_3$ following lemma from part 2 of solution, where we choose such a s that gives $r_2 - 2s$ equal to 0 or 1. After such an operation we can use part a or b.
So we have proved that whatever a_3 can be, numbers greater that $m \geq 2k + F(2, a_3)$ can be obtained from a_1, a_2, a_3 as a linear combination with non-negative coefficients, so $F(a_1, a_2, a_3) \leq 2k + F(2, a_3)$. Let's prove that $F(a_1, a_2, a_3) \geq 2k + F(2, a_3)$, when we'll get that these two numbers are equal. It's enough and needed to show that $a = 2k + F(2, a_3) - 1$ cannot be represented from a_1, a_2, a_3 as a linear combination with non-negative coefficients. $k = F(\frac{a_1}{2}, \frac{a_2}{2})$, when $a = 2k + a_3 - 2$. Let's assume that $2k + a_3 - 2 = r_1 a_1 + r_2 a_2 + r_3 a_3$, where $r_i \geq 0$. Left part is odd, so r_3 is odd too. Let's represent this equation following way: $2k - 2 = r_1 a_1 + r_2 a_2 + (r_3 - 1) a_3$. If $r_3 = 1$ when $2k - 2$ can be obtained as linear combination of a_1, a_2 with non-negative coefficients, but it was shown before that it cannot be. If $r_3 \geq 3$ when $(r_3 - 1) a_3 \geq 2k$ what gives that $r_1 a_1 + r_2 a_2 \leq -2$ that breaks the requirement of non-negativity of coefficients.

So we shown that any $s \geq 2F(\frac{a_1}{2}, \frac{a_2}{2}) + F(2, a_3)$ can be represented as linear combination of a_1, a_2, a_3 and $2F(\frac{a_1}{2}, \frac{a_2}{2}) + F(2, a_3) - 1$ cannot be, so $F(a_1, a_2, a_3) = 2F(\frac{a_1}{2}, \frac{a_2}{2}) + F(2, a_3)$.