Problem 1: Blocking sets

Team: Russia

Abstract

In this paper it was obtained that:

For point 1 of this problem in was found and proved that blocking set may have 8 points at minimum.

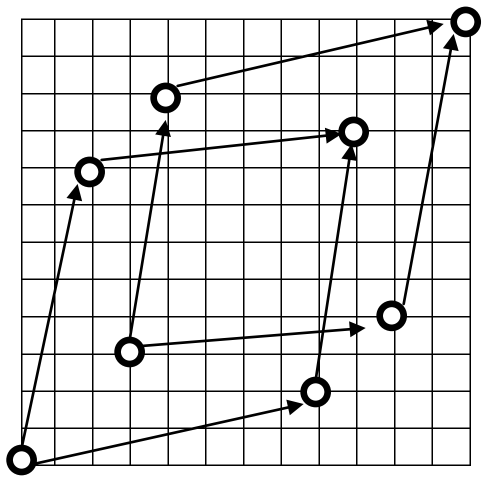
Also the optimal method for calculating D_n from point 3 was obtained. It may be calculated the following way:

$$D_n = \sum_{i=0}^{k} P(D(S) = i) * i,$$

1. What is the minimum number of elements that a blocking set can have?

The minimum number of elements that a blocking set can have is $eight$.

Let us make an example.



The coordinates of points are:

$$(0,0), (1/6,2/3), (2/3,1/6), (1/4,1/4), (5/6,1/3), (1/3,5/6), (3/4,3/4), (1,1).$$

There is no diagonal path in this graph.

Why 8 is the minimum number of elements that a blocking set can have?

There are 2 points in S.

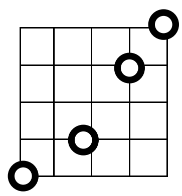
1) This means that S contains only $(0,0)$ and $(1,1)$. There is an arrow between these points, so there is a diagonal path when there are 2 points in S.

There are 3 points in S.

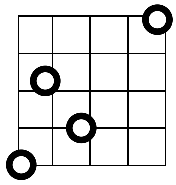
1) This means that S contains only one element (a, b) different from $(0,0)$ and $(1,1)$. There is an arrow from$(0,0)$ to (a, b) and there is an arrow from (a, b) to $(1,1)$, so there is a diagonal path when there are 3 points in S.

There are 4 points in S.

1) This means that S contains two elements (a, b) and (c, d) that are different from $(0,0)$ and $(1,1)$.
2) If there is only one arrow from $(0,0)$, for example, to (a, b), $c > a$ and $d > b$. So there is an arrow from (a, b) to (c, d) and there is an arrow from (c, d) to $(1,1)$. So there is a diagonal path.



3) If there are two arrows from $(0,0)$, then there are arrows from (a,b) to $(1,1)$ and from (c,d) to $(1,1)$. So there is a diagonal path.



4) So there is a diagonal path when there are 4 points in `S.

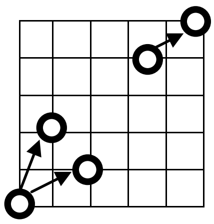
There is always at least one arrow from $(0,0)$, for example, to a point (a,b) and there is also at least one arrow to $(1,1)$, for example, from (c,d).

If there is only one arrow from $(0,0)$ to $(a,b), c > a$ and $d > b$. Because if this is not true and one of the coordinates of (c,d) is smaller than one of the coordinates of (a,b) , for example, $b > d$ it is not possible: $c < a, d < b$, because then there would be no arrow from $(0,0)$ to (a,b):

1) There would be an arrow from $(0,0)$ to (c,d), but this contradicts with the statement that there is only one arrow from $(0,0)$.
2) If there would not be an arrow from $(0,0)$ to (c,d), there would be another point that has second coordinate smaller than d, but has first coordinate bigger than a (so that there would be an arrow from $(0,0)$ to (a,b)), and has also the smallest second coordinate of all elements in S. That is why there would be an arrow from $(0,0)$ to this point and this contradicts with the statement that there is only one arrow from $(0,0)$.

If there is an arrow to $(1,1)$ from, for example, (e,f), we have to investigate situation when all points that have arrows from $(0,0)$ have smaller coordinates than (e,f). Because if one of them has one coordinate bigger (it can not have two coordinates bigger than (e,f) has because then there will be no arrow from (e,f) to $(1,1)$. Contradiction), it has an arrow to $(1,1)$ (the proof is the same as the previous statement), so there is a diagonal path.

That is why let's suppose that (e,f) has all coordinates bigger than coordinates of points that have arrows from $(0,0)$ to them.



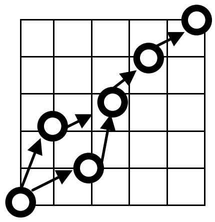
There are 5 points in S.

1) This means that S contains three elements (a,b) , (c,d) and (e,f) that are different from $(0,0)$ and $(1,1)$.
2) Let's suppose that the arrow to $(1,1)$ is from (e,f).
3) If there is only one arrow from $(0,0)$, for example, to (a,b) , $c > a$ and $d > b$ and there is an arrow from (a,b) to (c,d) and there is an arrow from (c,d) to (e,f). So there is a diagonal path.

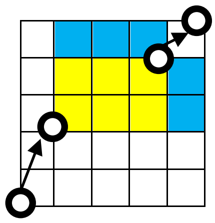
4) If there are two arrows from (0,0) to (a, b) and (e, f), there is a diagonal path and this situation would not be investigated again in other cases because it is simple.
5) If there are two arrows from (0,0) to (a, b) and (c, d), there is an arrow from (a, b) to (e, f)(if $a < e \text{ and } b < f$) or $(1,1)$ (if $a > e \text{ or } b > f$). So there is a diagonal path.
6) So there is a diagonal path when there are 5 points in S.

There are 6 points in S.

1) This means that S contains four elements (a, b) , (c, d), (e, f) and (g, h) that are different from $(0,0)$ and $(1,1)$.
2) Let's suppose that there is an arrow to (1,1) from (g, h).
3) There are two arrows from (0,0), for example, to (a, b) and (c, d). How to block the way from (a, b) to (g, h) and from (c, d) to (g, h) with only one point? There is only one way to do it: $e > a, e > c, f > b, f > d, e < g, f < h$. Arrows from (a, b) and (c, d) to (e, f) are blocked, but there are arrows from (a, b) to (e, f) and from (c, d) to (e, f) and there is also an arrow from (e, f) to (g, h). So there is a diagonal path.



4) There is only one arrow from (0,0), for example, to (a, b). How to block the way from (a, b) to (g, h) with two points? There are three ways of doing it. Points can be only in yellow and blue part because in other parts they will block arrows from (0,0) to (a, b) or from (g, h) to (1,1) or there will be two arrows from (0,0).



4.1) If there are two points in the yellow part, 1)there is one arrow from (a, b), for example, to (c, d), so there is an arrow from (c, d) to (e, f) and from (e, f) there is an arrow to (g, h), or 2) there are two arrows from (a, b) to (c, d) and (e, f) and from them there are arrows to (g, h), so there is a diagonal path.

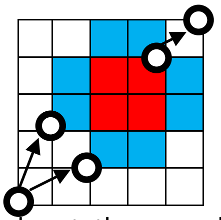
4.2) If there is only one point in yellow part, there would be an arrow from (a, b) to it and there also would be an arrow from it to (g, h).

4.3) If there are no arrows in yellow part, these two points would be in the same blue part (if it is not true and they are in different blue parts, the arrow from (g, h) to $(1,1)$ would be blocked. Contradiction with the statement that there is an arrow from (g, h) to $(1,1)$), so they would be an arrow from (a, b) to (g, h), so there would be a diagonal path.

5) So there is a diagonal path when there are 6 points in S.

There are 7 points in S.

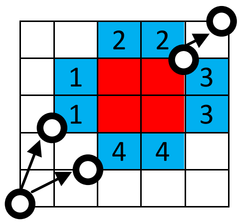
1) This means that S contains five elements (a, b), (c, d), (e, f), (g, h) and (i, j) that are different from (0,0) and (1,1).
2) Let's suppose that there is an arrow to $(1,1)$ $from$ (i, j).
3) There are two arrows from (0,0), for example, to (a, b) and (c, d). How to block the way from (a, b) to (i, j) and from (c, d) to (i, j) with two points? Red part is where even one point can do it. Blue part is where one point can block one arrow from (a, b) or (c, d) to (i, j). On the other parts of the board points will block arrows from (i, j) to $(1,1)$ or from $(0,0)$ to (a, b) and (c, d), or they will not block arrows from (a, b) and (c, d) to (i, j).



3.1) If there are two points in the red part, there would be an arrow from (a, b) or (c, d) to one of them, for example, to (e, f). And from this point there would be an arrow to (g, h), if $g > e, h > f$, or there would be an arrow to (i, j), if (g, h) is placed somewhere else. So there is a diagonal path.

3.2) If there is only one point in red part, for example, to (e, f), there would be an arrow from (a, b) or (c, d) to (e, f), because (g, h), which is in the blue part, will not block both arrows from (a, b) to (e, f) and from (c, d) to (e, f), because one of its coordinates would be bigger than the same coordinates of (a, b), (c, d) and (e, f). So there is a diagonal path.

3.3) If there are no points in the red part, there would be two points in the blue part. Let's number the blue sub-particles. Let's suppose that $a < c, b > d$.



3.4) If both points are in the same part, for example, in #1, they would not block one of the arrows from (a, b) or (c, d) to (i, j), because, in this situation, $d > f, d > h$. So there is a diagonal path.

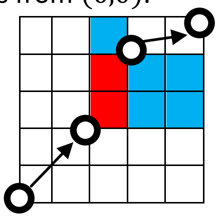
3.5) If one point is in 1, for example, (e, f) and the other is in 2 or 4, for example, (g, h), there would be an arrow from (a, b) to (e, f), from which there would be an arrow to (i, j). So there is a diagonal path.

3.6) If one point is in 1 and the other is in 3, there is an arrow from (c, d) to (i, j). So there is a diagonal path. The same situation when one of the points is in 2 and the other is in 4.

3.7) If one point is in 2 and the other is in 3, there is no arrow from $(i,j)\ to\ (1,1)$. Contradiction with the statement that there is an arrow from $(i,j)\ to\ (1,1)$.

3.8) If one of the points is in 3, for example, (e,f) and the other is in 4,for example, (g,h), there is an arrow from $(c,d)\ to\ (e,f)$, from which there is an arrow to (i,j). So there is a diagonal path.

4) There is only one arrow from $(0,0)$, for example, to (a,b). How to block the way from (a,b) to (i,j) with three points? The red part and the blue part is where these three points can be. Because if they are in other parts, they would block arrows from $(0,0)\ to\ (a,b)$ or from (i,j) to $(1,1)$ or there would be two arrows from $(0,0)$.



4.1) If there are points in the blue part, they would be in one blue sub-part. Because if they would be in different sub-parts, they would block the arrow from $(i,j)\ to\ (1,1)$.
4.2) If there are no points in the red part, there would be an arrow from $(a,b)\ to\ (i,j)$. So there is a diagonal path.
4.3) If there is only one point in the red part, there would be an arrow from (a,b) to this point, because other two points, which are in the same blue sub-part, have one coordinate not between same coordinates of (a,b) and of the point that is in the red part. And from this point there would be an arrow to (i,j), because of the same reason. So there is a diagonal path.
4.4) If there are two points in the red part, there would be an arrow from (a,b) to one of the points in the red part, for example, to (c,d). And from (c,d) there would be an arrow to the other point in the red part, for example, to (e,f), if $e > c\ and\ f > d$, from which there would be an arrow to (i,j). Or if there would be an arrow to (i,j), if (e,f) is placed somewhere else. So there is a diagonal path.
4.5) If there are three points in the red part, the situation would be similar to the case, when $a = 0, b = 0, i = 1, j = 1$ and there are 5 points in S. So there is a diagonal path.

5) So there is a diagonal path when there are 7 points in S.

That is why 8 is the minimum number of elements that a blocking set can have.

3. Calculating an arithmetic mean of $D(S)$ for an n-point set.

Let $D(S)$ be a number of diagonal paths in n-point set S. And let D_n be an ariphmetic mean of $D(S)$ for all n-point acceptable sets. Let us calculate D_n. If all $D(S)$ were equeprobable for a n-point set S, then D_n would be calculated this way:

$$D_n = \frac{\sum_i D(S)}{k}, where\ k\ is\ a\ number\ of\ possible\ numbers\ of\ diagonal\ paths, in\ n-point\ set\ S$$

But, we know that $D(S)$ are not equiprobable, so we may consider the probability theory kind of arithmetic mean of a sequence. It is:

$$A_n = \sum_i P(A_i) * k(A_i),$$

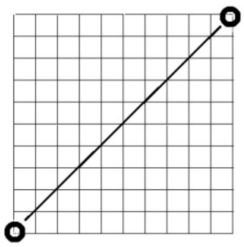
where A_i is an element of a sequence, $P(A_i)$ is probability of A_i and $k(A_i)$ is meaning of A_i

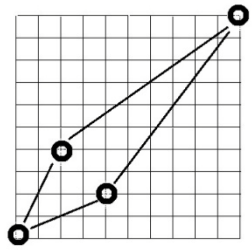
So we may consider our D_n in terms of this definition and write it the following way:

$$D_n = \sum_{i=0}^{k} P(D(S) = i) * i,$$

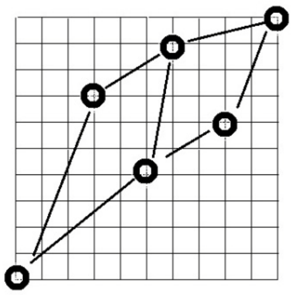
Where k is a maximum probable number of diagonal paths of an n-point set S. Also it is obvious, that we can count all the numbers for all sets, but it can be found that big numbers of diagonal paths for a certain n is impossible, so that $P(D(S) = t) = 0$, where t is rather big for a certain n, so we can count only numbers that are lower or equal to maximal for a certain n.

Let us firstly estimate k, maximum possible number of diagonal paths in an n-point set S. Let us consider a 2-point and 4-point sets.





2-point set has 1 maximum possible diagonal, that is obvious and 4-point set may have 2 maximum diagonal paths, then let us consider a set that may have maximum 3 diagonal paths. It is a 6-point set.

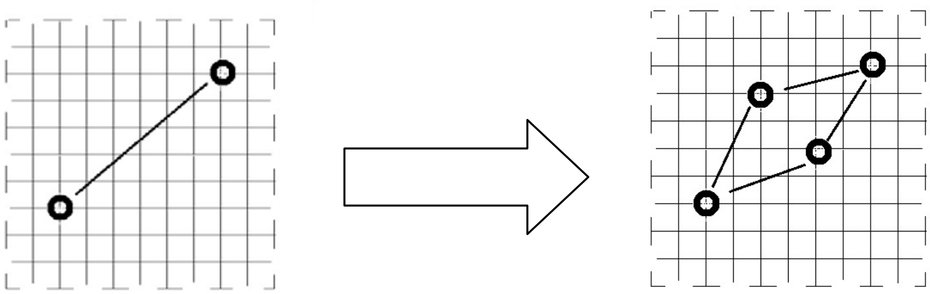


So we may make an assumption:

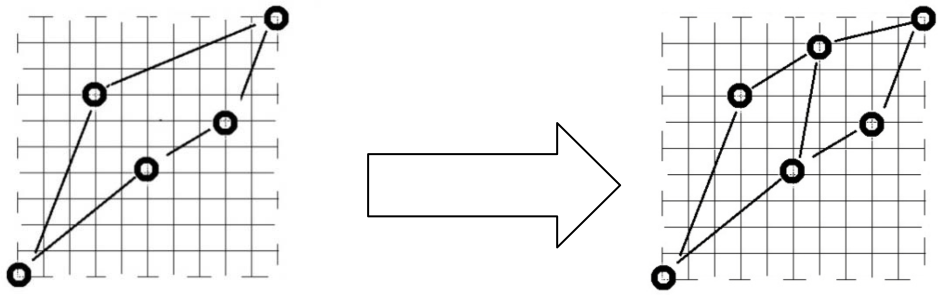
If we have an n-point set, where $n >= 4$ that may have k diagonal paths at maximum, then we have to add 2 more points to get a set that may have $k + 1$ diagonal paths at maximum.

Let us look at the following case:

Let us consider case of maximal number of diagonal paths and look at one of arrows.



If we put a new point between two “old” arrows it would also make one new arrow.



But adding one arrow doesn’t mean adding one diagonal path, it means multiplying some number of paths by two. And namely, all that paths, that contained an “old” arrow. So we can say, that adding two points may only add one arrow and may multiply number of all paths by two, so the assumption is wrong, but it means that another assumption is right:

$$k = 2^{\frac{n}{2}-2}\ for\ odd\ n, and\ k = 2^{\frac{n-1}{2}-2}$$

That means that we have found upper border for summing. So, we only have to calculate all the probabilities of getting i diagonal paths in an n-point set and D_n will be calculated.