Problem 9: A Topological Problem

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Abstract

We are giving complete solutions for all the proposed problems. Our answers are: You can generate 7 sets with the operations int and cl, and with the operations int, cl and conv you can generate 6 sets if the starting set is non-essential — i.e., satisfies $\operatorname{int}\operatorname{conv} A = \emptyset$ — (2 sets for $n = 1$; 5 sets for $n = 2$) and 17 sets if the starting set is essential (14 sets for $n = 1$).

To prove these results, we prove a long series of lemmas providing identities between the operators int, conv and cl that hold for every dimension n (up to some exceptions which are explicitely marked). We have first proved all the lemmas and then begun to answer the problems because most of the lemmas help in several parts of the problem, and there are also some lemmas whose proofs depend on other lemmas.

Most of the proofs are rather technical. There is only one remarkable idea: namely to use the help of halfplanes — these are sets of the form

$$H(y, n) := \{x \in \mathbb{R}^n; (x - y)n > 0\}$$

— to describe convex sets. The main lemma linking halfplanes and convex sets is Lemma 5. The proof of this lemma is the most technical of all proofs, but having proved this lemma, halfplanes become a powerful tool to shorten the proofs of the following lemmas. The reason for this is that if the arguments are halfplanes, there are many easy relations between the three operators int, cl and conv: If H is a halfplane, then $\operatorname{int} H = \operatorname{conv} H = \operatorname{int}\operatorname{cl} H = H$, and $\operatorname{conv}\operatorname{cl} H = \operatorname{cl} H$. Most of these are 'inherited', in one form or the other, to convex sets.

We will first prove some general lemmas on the operators cl, int and conv valid for all n. In the following, let n be a positive integer and let A and B denote subsets of $\mathbb{R}^n$.

Lemma 1. We have $\operatorname{int} A \subseteq A$, $A \subseteq \operatorname{cl} A$ and $A \subseteq \operatorname{conv} A$. The sets $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{conv} A$ are open, closed and convex respectively. If $A \subseteq B$, then $\operatorname{int} A \subseteq \operatorname{int} B$, $\operatorname{cl} A \subseteq \operatorname{cl} B$ and $\operatorname{conv} A \subseteq \operatorname{conv} B$. The set A is open, closed, convex if and only if $\operatorname{int} A = A$, $\operatorname{cl} A = A$ and $\operatorname{conv} A = A$ respectively.

Proof. The proofs of all statements are immediate from the definitions. □

Lemma 2. a) If A is open, then $\operatorname{cl}\operatorname{int}\operatorname{cl} A = \operatorname{cl} A$.
b) If A is closed, then $\operatorname{int}\operatorname{cl}\operatorname{int} A = \operatorname{int} A$.

Proof. a) From Lemma 1 it follows that $\operatorname{int}\operatorname{cl} A \subseteq \operatorname{cl} A$ and thus $\operatorname{cl}\operatorname{int}\operatorname{cl} A \subseteq \operatorname{cl}\operatorname{cl} A = \operatorname{cl} A$.
Also, as A is open, we have $\operatorname{int} A = A$, so $\operatorname{int}\operatorname{cl} A \supseteq \operatorname{int} A = A$ and, consequently, $\operatorname{cl}\operatorname{int}\operatorname{cl} A \supseteq \operatorname{cl} A$.
b) We trivially have $\operatorname{int} A \subseteq \operatorname{cl}\operatorname{int} A$, so $\operatorname{int} A = \operatorname{int}\operatorname{int} A \subseteq \operatorname{int}\operatorname{cl}\operatorname{int} A$.
Moreover, as A is closed, we have $A = \operatorname{cl} A \supseteq \operatorname{cl}\operatorname{int} A$, so $\operatorname{int} A \supseteq \operatorname{int}\operatorname{cl}\operatorname{int} A$. □

Lemma 3. a) If A is convex, then $\operatorname{int} A$ is convex.
b) If A is convex, then $\operatorname{cl} A$ is convex.
c) If A is open, then so is $\operatorname{conv} A$.

Proof. a) Suppose that $\operatorname{int} A$ is not convex. Then there are $x, y \in \operatorname{int} A$ and a point z on the line segment $[x, y]$ such that $z \notin \operatorname{int} A$. Then[1] there is a sequence (z_i) converging to z such that $z_i \notin A$ for all i. Let $z = \lambda x + (1-\lambda) y$ with $\lambda \in [0, 1]$. Obviously, λ is different from 0 and 1 as $z \neq x, y$. For all $i \in \mathbb{N}$, let $x_i = \frac{z_i}{\lambda} - \frac{1-\lambda}{\lambda} y$. Then we always have $x_i \notin A$ as otherwise, z_i would have to lie in A because $z_i = \lambda x_i + (1-\lambda) y$ lies on the segment $[x_i, y]$. For $i \to \infty$, we have $z_i \to z$ and thus $x_i \to \frac{z}{\lambda} - \frac{1-\lambda}{\lambda} y = x$, so there is a sequence of points which are not in A converging to x, contradicting the fact that $x \in \operatorname{int} A$.
b) Suppose the contrary. Then there are $x, y \in \operatorname{cl} A$ and a point z on the line segment $[x, y]$ such that $z \notin \operatorname{cl} A$. As $x, y \in \operatorname{cl} A$, there are sequences $(x_n), (y_n)$ converging to x, y such that $x_n, y_n \in A$ for all n. Now, write

[1] Here we use the fact that a subset A of $\mathbb{R}^n$ is closed if and only if for every sequences of points in A that converges to some point a, the point a also lies in a. This is a well-known fact. In the sequel, we will use this and similar known facts about the topology of $\mathbb{R}^n$ without further mentioning.

$z = \lambda x + (1-\lambda)y$ with $\lambda \in [0,1]$. Then we have $z_n := \lambda x_n + (1-\lambda)y_n \in A$ for all n by the convexity of A, and as $n \to \infty$, the z_n tend to z. It follows that $z \in \operatorname{cl} A$.
c) If A is open, then $\operatorname{int} A = A$. As $A \subseteq \operatorname{conv} A$, we have $A = \operatorname{int} A \subseteq \operatorname{int}\operatorname{conv} A$. Furthermore, $\operatorname{int}\operatorname{conv} A$ is convex by a). Thus, $\operatorname{int}\operatorname{conv} A$ is a convex set containing A, so $\operatorname{conv} A \subseteq \operatorname{int}\operatorname{conv} A$. Now, $\operatorname{int}\operatorname{conv} A \subseteq \operatorname{conv} A$ holds trivially, so we get that $\operatorname{conv} A = \operatorname{int}\operatorname{conv} A$ and hence, $\operatorname{conv} A$ is open. □

An $(n-1)$-dimensional hyperplane is a set of the form $E(y,n) = \{x \in \mathbb{R}^n; (x-y)n = 0\}$. It is easy to see that $(n-1)$-dimensional hyperplanes are convex and closed and have empty interior.

Defintion 1. We call a set $A \subseteq \mathbb{R}^n$ *essential* if it's not contained in an $(n-1)$-dimensional hyperplane.

Lemma 4. A is essential if and only if $\operatorname{int}\operatorname{conv} A = \emptyset$.

Proof. If A is not essential, say if A is contained in the hyperplane E, then it follows that $\operatorname{int}\operatorname{conv} A \subseteq \operatorname{int}\operatorname{conv} E = \operatorname{int} E = \emptyset$.
Now, let A be essential. Then there are n points $A_1, \dots A_{n+1} \in A$ which are not contained in an $(n-1)$-dimensional hyperplane; It follows that every point in $\mathbb{R}^n$ has exactly one representation as $\sum_{i=1}^{n+1} \varepsilon_i A_i$ with $\sum \varepsilon_i = 1$. (This can be seen as follows: Suppose without loss of generality that $A_1 = 0$. Then the condition that $A_1, \dots A_{n+1}$ are not on an $(n-1)$-dimensional hyperplane is equivalent to saying that $A_2, \dots A_{n+1}$ are linearly independent. Thus, every point in the $\mathbb{R}^n$ can uniquely be written as $\sum_{i=2}^{n+1} \varepsilon_i A_i$; adding a summand $\varepsilon_1 A_1$ doesn't change anything in this representation, so the unique choice for ε_1 such that $\sum \varepsilon_i = 1$ gives a unique solution to the problem; it is easy to transfer this to the case that $A_1 \neq 0$, but we can also assume $A_1 = 0$ for the rest of the proof).
It is easy to see that the set Δ of all the points $\sum_{i=1}^{n+1} \varepsilon_i A_i$ with $\sum \varepsilon_i = 1$ and $\varepsilon_i \geq 0$ for all i are contained in $\operatorname{conv} A$. We claim that there is an open ball around $S := \frac{\sum A_i}{n}$ that is contained in Δ. Suppose the contrary. Then there is a sequence of points $S_i = \sum_{j=1}^{n+1} \varepsilon_{ij} A_j$, $\sum \varepsilon_{ij} = 1$ for fixed i, converging to S such that $S_i \notin \Delta$. This means that always $\varepsilon_{ij} < 0$ for one j; among all these js, one has to occur infinitely often, and by passing to a subsequence we can assume that $\varepsilon i1 < 0$ for all i. But taking the first coordinate in a representation $X = \sum_{i=1}^{n+1} \varepsilon_i A_i$ with $\sum \varepsilon_i = 1$ is surely continuous in X, so as $S_i \to S$, we get that $\frac{1}{n} \leq 0$, contradiction.
Thus, $\Delta \subseteq \operatorname{conv} A$ contains an open ball and thus $\operatorname{int}\operatorname{conv} A$ is non-empty. □

Defintion 2. An open halfplane (or just halfplane) is a set of the form

$$H(y,n) := \{x \in \mathbb{R}^n; (x-y)n > 0\}$$

with $y, n \in \mathbb{R}^n$, where the product in the inequality is the scalar product. The boundary of the halfplane $H(y,n)$ is defined as

$$\partial H(y,n) = \{x \in \mathbb{R}^n; (x-y)n = 0\}.$$

It is easy to check that halfplanes are open and convex, and that $\operatorname{cl} H = H \cup \partial H$ for any halfplane H.

Lemma 5. a) If $A \subseteq \mathbb{R}^n$ is closed and convex, then A can be written as an intersection of closed halfplanes, i.e. there are an index set I and halfplanes $H_i, i \in I$, such that $A = \bigcap_{i \in I} \operatorname{cl} H_i$.
b) If $A \subseteq \mathbb{R}^n$ is convex and $x \notin A$, then there is a halfplane H such that $x \in \partial H$ and $A \subseteq \operatorname{cl} H$.
c) If $A \subseteq \mathbb{R}^n$ is open and convex, then A can be written as an intersection of open halfplanes.

Proof. a) We prove that for every $x \notin A$, there is a halfplane H_x such that $x \notin \operatorname{cl} H_x$ and $A \subseteq \operatorname{cl} H_x$. Then it follows easily that $A = \bigcap_{x \notin A} \operatorname{cl} H_x$.
Let $x \notin A$. Let d be the distance from x to A. We have $d > 0$ as A is closed. Now, let A' be the set of all $x \in \mathbb{R}^n$ such that there is an $a \in A$ with $d(x,a) \leq \frac{d}{2}$. Then A' is closed and convex (this follows immediately from the fact that A is closed and convex), and we have $x \notin A'$. We will show that there is a halfplane H such that $x \in \partial H$ and $A' \subseteq H$. From this, it is easy to construct a halfplane H' such that $x \notin \operatorname{cl} H'$ and $A \subseteq H'$. (If $H = H(y,n)$ with $|n| = 1$, then $H' = H(y + \varepsilon n, n)$ will do.)
Furthermore, we can assume that A' is compact: Suppose that we have shown our claim for every compact set. Then setting $A'_n := A_n \cap \operatorname{cl} B_n(0)$ for every n, all the A'_n are compact and their union is A'. Thus, we can choose halfplanes H_n such that $x \in \partial H_n$ and $A'_n \subseteq \operatorname{cl} H_n$. Let $H_n = H(x, u_n)$ with $|u_n| = 1$. Then the u_n are elements of $S^{n-1} = \{y \in \mathbb{R}^n; |y| = 1\}$, and this is compact, so there is a subsequence (u_{n_i}) of the sequence (u_n) converging to $u \in S^{n-1}$. Then the halfplane $H(x,u)$ satisfies $x \in \partial H(x,u)$ and $A' \subseteq \operatorname{cl} H(x,u)$. (To see why the latter holds, fix $a \in A'$. Then there is $i_0 \in \mathbb{N}$ such that $|a| < n_{i_0}$ $a \in \operatorname{cl} H_{n_i}$ for $i > i_0$. This gives $(a-x)u_{n_i} > 0$ for $i > i_0$. Letting $i \to \infty$, it follows that $(a-x)u > 0$.)
Without loss of generality, let $x = 0$. We 'project' the set A onto the unit sphere $S = S^{n-1} = \{y \in \mathbb{R}^n; |y| = 1\}$ by setting

$$U := \{y \in S; \text{there is } \lambda > 0 \text{ such that } \lambda y \in A'\}.$$

It is obvious that U can't contain any two points which are situated on a diameter of S, so there is a point $y \notin U$. We can choose y such that there is a little environment of y disjoint with U. (The latter statement is trivial if A is not essential. If A is essential, then $\operatorname{int} A = \operatorname{int}\operatorname{conv} A \neq \emptyset$, so there is an open ball $O \subseteq A$, and the 'projection' of O onto S will give an open environment on S which is contained in U. Reflect this at the origin to get an environment which is disjoint with U.) Now, consider $R := \{r \in \mathbb{R}; \forall z \in S : yz \geq r \Rightarrow z \notin U\}$. By the choice of y, R contains a neighbourhood of 1. Let $r_0 := \inf R < 1$. If $r_0 \leq 0$ then A' is contained in the closed halfplane given by $yz < 0$, so we are done. So let's assume that $r_0 > 0$. Let r_i be a sequence of points converging to r_0 from below such that $r_i \notin R$ for all i. Then there are $z_i \in U$ with $r_i \leq yz_i \leq r_0$ for all z_i. As S is compact, the sequence (z_i) has a converging subsequence, and we can assume w.l.o.g. that (z_i) converges to $z \in S$. Then the z_i have multiples $\hat{z}_i$ in A', and as A' is compact, the sequence $(\hat{z}_i)$ has a subsequence converging to some point $\hat{z} \in A'$. As the projection onto S is continuous, it follows that $\hat{z}$ is a multiple of z, so $z \in U$.
We clearly have $yz = r$, as $r_i \to r$. Now, let

$$H := \{t \in \mathbb{R}^n; (y - r_0 z)t > 0\}.$$

Then we have $A' \cap H = \emptyset$, so that A' is contained in the complement of H, which is the closure of a halfplane. To see that A' and H are disjoint, suppose that $a \in A' \cap H$ and let $b = \frac{a}{|a|}$. Then for $s_0 := by$ we get that $s_0 > r_0 bz$. We find real numbers μ_1, μ_2 such that $\mu_1 z, \mu_2 b \in A'$. Then for every $\lambda \in [0,1]$, $\lambda\mu_1 z + (1-\lambda)\mu_2 b \in A'$, so

$$f(\lambda) := \frac{y(\lambda\mu_1 z + (1-\lambda)\mu_2 b)}{|\lambda\mu_1 z + (1-\lambda)\mu_2 b|} < r_0.$$

But a short calculation gives $f(1) = r_0$ and $f'(1) = r_0 bz - s_0 < 0$, so $f(\lambda) > r_0$ if λ is < 1, but close enough to 1. Contradiction!
b) Note that $\operatorname{cl} A$ is convex and closed. Thus if $x \notin \operatorname{cl} A$, then there is a hyperplane H with $x \notin H$ and $A \subseteq \operatorname{cl} H$. By translating H, we can get that $x \in \partial H$ and still $A \subseteq \operatorname{cl} H$.
Now, let's consider the case that $x \in \operatorname{cl} A$. As before, assume that $x = 0$. Define U as in a). Similarly to a), it can be shown that U is not dense in S. It follows that there is a $y \in S$ such that every multiple of y has a neighbourhood disjoint with A and thus is outside $\operatorname{cl} A$, so there is a sequence x_i converging to x such that $x_i \notin \operatorname{cl} A$. By the above, we can choose halfplanes H_i such that $x_i \in \partial H_i$ and $A \subseteq \operatorname{cl} H_i$. Let H_i be given by the inequality

$(t - x_i)n_i > 0$ where we choose n_i such that $|n_i| = 1$. Then all the n_i are vectors from S, so as S is compact, (n_i) has a converging subsequence and we can assume w.l.o.g. that the n_i converge to some $n \in S$. Let H be the halfspace given by the inequality $(t - x)n > 0$. Then obviously, $x \in \partial H$. Moreover, we have $A \subseteq \operatorname{cl} H$: Let $a \in A$. Then $(a - x_i)n_i \geq 0$ for all i. Letting $i \to \infty$, we get that $(a - x)n \geq 0$.
c) Let $x \notin A$. By b), there exists a halfplane H_x such that $x \in \partial H_x$ and $A \subseteq \operatorname{cl} H_x$. As A is open, it follows that $A \subseteq \operatorname{int} \operatorname{cl} H_x = H_x$. So we have $x \notin H_x$ and $A \subseteq H_x$. Now it's easy to see that $A = \bigcap_{x \notin A} H_x$. □

Lemma 6. We have $\operatorname{cl} \operatorname{conv} \operatorname{cl} A = \operatorname{cl} \operatorname{conv} A$.

Proof. As $A \subseteq \operatorname{cl} A$, it is clear that $\operatorname{cl} \operatorname{conv} A \subseteq \operatorname{cl} \operatorname{conv} \operatorname{cl} A$.
As for the other inclusion, let $x \notin \operatorname{cl} \operatorname{conv} A$. As $\operatorname{cl} \operatorname{conv} A$ is closed and convex by Lemma 3b, we can apply Lemma 5a and get that there is a halfplane H such that $x \notin \operatorname{cl} H$ and $\operatorname{cl} \operatorname{conv} A \subseteq \operatorname{cl} H$. As $A \operatorname{cl} \operatorname{conv} A$, we get that $A \subseteq \operatorname{cl} H$ and thus $\operatorname{cl} \operatorname{conv} \operatorname{cl} A \subseteq \operatorname{cl} \operatorname{conv} \operatorname{cl} \operatorname{cl} H = \operatorname{cl} H$. Thus, $x \notin \operatorname{cl} \operatorname{conv} \operatorname{cl} H$. □

Lemma 7. We always have $\operatorname{int} \operatorname{conv} \operatorname{cl} A = \operatorname{int} \operatorname{conv} A$.

Proof. It is obvious that $\operatorname{int} \operatorname{conv} \operatorname{cl} A \supseteq \operatorname{int} \operatorname{conv} A$. For the other inclusion, we distinguish two cases:
Case 1: A is essential. Let $x \notin \operatorname{int} \operatorname{conv} A$. Then by Lemma 5c, there is a halfplane H such that $x \notin H$ and $\operatorname{int} \operatorname{conv} A \subseteq H$. As $\operatorname{int} \operatorname{conv} A$ is open and not empty, there is an open ball O contained in $\operatorname{int} \operatorname{conv} A$ (and thus also in H). I now claim that $\operatorname{conv} A \subseteq \operatorname{cl} H$. To prove the claim, suppose the contrary. Then there is a point $a \in \operatorname{conv} A$ with $A \notin \operatorname{cl} H$. Without loss of generality, suppose that $a = 0$. As $O \subseteq \operatorname{conv} A$ and $\operatorname{conv} A$ is open, the set λO will be contained in $\operatorname{conv} A$ for all $\lambda \in [0, 1]$. It is obvious that λO will be disjoint from $\operatorname{cl} H$ for λ small enough: Let H be given by the equation $(x - y)n > 0$. Then $-yn < 0$ or, equivalently, $yn > 0$, as $0 \notin \operatorname{cl} H$. Now, O is bounded, so we have $x \leq C$ for all $x \in O$. Let $0 < \lambda < \frac{yn}{2C|n|}$ (this is possible as $yn > 0$). Then for $x \in O$, we have, by the Cauchy-Schwarz inequality, $|\lambda xn| < \lambda |x||n| < \lambda C|n| < \frac{yn}{2}$, so

$$(\lambda x - y)n = \lambda xn - yn < |\lambda xn| - yn < \frac{yn}{2} - yn = -\frac{yn}{2} < 0,$$

and hence $\lambda x \notin \operatorname{cl} H$. Thus, for λ small enough, the open set λO is disjoint with H. But λO is an open ball and it's contained in $\operatorname{conv} A$, so it's also contained in $\operatorname{int} \operatorname{conv} A$. But this contradicts the fact that $\operatorname{int} \operatorname{conv} A \subseteq H$.

Thus, our claim that $\operatorname{conv} A \subseteq \operatorname{cl} H$ is proved. From this, we get that $A \subseteq \operatorname{conv} A \subseteq \operatorname{cl} H$, so $\operatorname{int} \operatorname{conv} \operatorname{cl} A \subseteq \operatorname{int} \operatorname{conv} \operatorname{cl} \operatorname{cl} H = H$, and hence $x \notin \operatorname{int} \operatorname{conv} \operatorname{cl} A$.

Case 2: A is not essential. Then A is contained in some hyperplane E. It follows that $\operatorname{int} \operatorname{conv} \operatorname{cl} A \subseteq \operatorname{int} \operatorname{conv} \operatorname{cl} E = \operatorname{int} E = \emptyset$. □

Lemma 8. If A is convex, then $\operatorname{int} \operatorname{cl} A = \operatorname{int} A$.

Proof. By Lemma 3b, $\operatorname{cl} A$ is also convex, so by Lemma 7,

$$\operatorname{int} \operatorname{cl} A = \operatorname{int} \operatorname{conv} \operatorname{cl} A = \operatorname{int} \operatorname{conv} A = \operatorname{int} A.$$

□

Lemma 9. If A is open, then $\operatorname{conv} \operatorname{int} \operatorname{cl} A = \operatorname{conv} A$.

Proof. We have $\operatorname{cl} A \supseteq A$. As A is open, it follows that $\operatorname{int} \operatorname{cl} A \supseteq A$ and hence $\operatorname{conv} \operatorname{int} \operatorname{cl} A \supseteq \operatorname{conv} A$.

Now, let $x \notin \operatorname{conv} A$. By Lemma 5c, there is a hyperplane H such that $x \notin H$ and $\operatorname{conv} A \subseteq H$. It follows that $A \subseteq \operatorname{conv} A \subseteq H$, so $\operatorname{conv} \operatorname{int} \operatorname{cl} A \subseteq \operatorname{conv} \operatorname{int} \operatorname{cl} H = H$. Consequently, $x \notin H$. □

Lemma 10. If A is convex and essential, then $\operatorname{cl} \operatorname{int} A = \operatorname{cl} A$.

Proof. It is obvious that $\operatorname{cl} \operatorname{int} A \subseteq \operatorname{cl} A$. For the other direction, let $x \notin \operatorname{cl} \operatorname{int} A$. As $\operatorname{cl} \operatorname{int} A$ is closed and convex, there is a halfplane H such that $x \notin \operatorname{cl} H$ and $\operatorname{cl} \operatorname{int} A \subseteq \operatorname{cl} H$. It follows that $\operatorname{int} A \subseteq H$ and as in Case 1 of the proof of Lemma 7, it can be shown from this that $A \subseteq \operatorname{cl} H$. Thus, $\operatorname{cl} A \subseteq \operatorname{cl} H$ and $x \notin \operatorname{cl} A$. □

Lemma 11. If $n = 1$, then $\operatorname{cl} \operatorname{conv} A = \operatorname{conv} \operatorname{cl} A$.

Proof. Let $a = \inf A$ and $b = \sup A$. Assume that $a, b \neq \pm\infty$ (the cases that one of a, b equals $+\infty$ or $-\infty$ are treated very similarly).

Choose sequences $a_i \to a, b_i \to b$ with $a_i, b_i \in A$. Then $[a_i, b_i] \subseteq \operatorname{conv} A$ for all i and it follows that $(a, b) \subseteq \operatorname{conv} A$ and thus $[a, b] \subseteq \operatorname{cl} \operatorname{conv} A$. On the other hand, $[a, b]$ is convex and closed and $A \subseteq [a, b]$, so it follows that $\operatorname{conv} A \subseteq [a, b]$ and thus $\operatorname{cl} \operatorname{conv} A \subseteq [a, b]$. Thus, $\operatorname{cl} \operatorname{conv} A = [a, b]$.

Now, we have $a, b \in \operatorname{cl} A$ and thus $[a, b] \subseteq \operatorname{conv} \operatorname{cl} A$. As above, it's easy to see that $\operatorname{conv} \operatorname{cl} A \subseteq [a, b]$; thus, $\operatorname{conv} \operatorname{cl} A = [a, b] = \operatorname{cl} \operatorname{conv} A$. □

Lemma 12. Let $A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m$; consider $A \times B$ as a subset of $\mathbb{R}^{m+n}$; then
a) $\operatorname{int}(A \times B) = \operatorname{int} A \times \operatorname{int} B$,
b) $\operatorname{cl}(A \times B) = \operatorname{cl} A \times \operatorname{cl} B$,
c) $\operatorname{conv}(A \times B) = \operatorname{conv} A \times \operatorname{conv} B$.

Proof. The proofs of a) and b) are very easy. To prove c), it is easy to see that $\operatorname{conv} A \times \operatorname{conv} B$ is convex, and as $A \subseteq \operatorname{conv} A$ and $B \subseteq \operatorname{conv} B$ it follows that $A \times B \subseteq \operatorname{conv} A \times \operatorname{conv} B$ and thus $\operatorname{conv}(A \times B) \subseteq \operatorname{conv} A \times \operatorname{conv} B$. Now, let's prove that $\operatorname{conv}(A \times B) \supseteq \operatorname{conv} A \times \operatorname{conv} B$. Fix $b \in B$. Then let $A' = \{a \in \mathbb{R}^n; (a, b) \in \operatorname{conv}(A \times B)\}$. We have $A \subseteq A'$ and A' is convex (for $a_1, a_2 \in A'$ and $\lambda \in [0, 1]$, we have $(\lambda a_1 + (1 - \lambda)a_2, b) = \lambda(a_1, b) + (1 - \lambda)(a_2, b) \in \operatorname{conv}(A \times B)$ as $\operatorname{conv}(A \times B)$ is convex), so $\operatorname{conv} A \subseteq A'$. We get that $\operatorname{conv} A \times \{b\} \subseteq \operatorname{conv}(A \times B)$ for any $b \in B$ and thus $\operatorname{conv} A \times B \subseteq \operatorname{conv}(A \times B)$. By a similar argument as above, $\operatorname{conv} A \times \operatorname{conv} B \subseteq \operatorname{conv}(A \times B)$.

□

Now we will begin to solve the problems. We solve problems 9.1 and 9.2 for any dimension, thus also solving problem 9.3.

Problem 9.1

Answer: The maximal number is 7.

Proof. Why can't we obtain more than seven sets?
We can always only obtain the seven sets $A, \operatorname{int} A, \operatorname{cl} \operatorname{int} A, \operatorname{int} \operatorname{cl} \operatorname{int} A, \operatorname{cl} A, \operatorname{int} \operatorname{cl} A$ and $\operatorname{cl} \operatorname{int} \operatorname{cl} A$. More sets (different from the sets obtained before) are not possible because $\operatorname{int} \operatorname{int} X = \operatorname{cl} \operatorname{cl} X = X$ for every X as well as $\operatorname{int} \operatorname{cl} \operatorname{int} \operatorname{cl} A = \operatorname{int} \operatorname{cl} A$ and $\operatorname{cl} \operatorname{int} \operatorname{cl} \operatorname{int} A = \operatorname{cl} \operatorname{int} A$ by Lemma 2a, b.

Why can we obtain seven sets?
Define the following subsets of $\mathbb{R}^n$: Let D^* be an open ball with the point O in its center removed, D_{rat} the intersection of an open ball with $\mathbb{Q}^n$ and $\{P\}$ a one-point set. The closure of D^* is a closed ball C_1 whose interior is $D_1 = D^* \cup \{O\}$; the interior of D^* is again D^* as D^* is open. The closure of D_{rat} is a closed ball C_2; the interior of C_2 is an open ball D_2. The interior of $\{P\}$ is empty, the closure of $\{P\}$ is $\{P\}$.
Suppose that D^*, D_{rat} and P are positioned in $\mathbb{R}^n$ such that their closures are disjoint. Let $A = D^* \cup D_{\text{rat}} \cup \{P\}$. Then one easily sees that

$$\operatorname{int} A = D^*, \quad \operatorname{cl} \operatorname{int} A = C_1, \quad \operatorname{int} \operatorname{cl} \operatorname{int} A = D_1,$$

$$\operatorname{cl} A = C_1 \cup C_2 \cup \{P\}, \quad \operatorname{int}\operatorname{cl} A = D_1 \cup D_2, \quad \operatorname{cl}\operatorname{int}\operatorname{cl} A = C_1 \cup C_2,$$

yielding seven different sets generated by A.
Figure 1 shows the set A and the seven sets generated by A in the case $n = 2$.
□

Problem 9.2

a) This is Lemma 7 (it is valid for arbitrary n and also if A is not essential).

b) **Answer:** We can obtain two sets if A is not essental and $n = 1$, five sets if A is not essential and $n = 2$, six sets if A is not essential and $n \geq 3$, 14 sets if A is essential and $n = 1$ and 17 sets if n is essential and $n \geq 2$.

Proof for the non-essential case: Suppose that A is not essential. Then A is contained in an $(n-1)$-dimensional hyperplane E. As $\operatorname{int} E$, $\operatorname{cl} E$ and $\operatorname{conv} E$ are contained in E, all the sets that we get from A will also be non-essential. As the interior of a non-essential set is always empty, we will get at most one set (the empty set) more than we can generate from A with the operations cl and conv. Note that this is an $(n-1)$-dimensional problem: We can send E to the $\mathbb{R}^{n-1}$ by a bijective linear mapping (which is also a homeomorphism), take the closure (the convex hull) of the image of A in this $\mathbb{R}^{n-1}$ and then send it back to E and we get the closure (the convex hull) of A as a subset of $\mathbb{R}^n$. The reason for this is that the intersection of two closed (convex) sets is closed (convex) and that E is closed (convex), so that for every closed (convex) set B containing A, there is another closed convex set $C = B \cap E$ such that $A \subseteq B \subseteq C \subseteq E$.
Thus, the question is how many sets one can generate from an $(n-1)$-dimensional set with the operations cl and conv. We will always get no more than the five sets $A, \operatorname{cl} A, \operatorname{conv}\operatorname{cl} A, \operatorname{conv} A$ and $\operatorname{cl}\operatorname{conv} A$ as $\operatorname{cl}\operatorname{cl} = \operatorname{cl}$, $\operatorname{conv}\operatorname{conv} = \operatorname{conv}$, $\operatorname{cl}\operatorname{conv}\operatorname{cl} = \operatorname{cl}\operatorname{conv}$ by Lemma 6 and $\operatorname{conv}\operatorname{cl}\operatorname{conv} = \operatorname{cl}\operatorname{conv}$ by Lemma 3b.
If $n = 1$, then the $(n-1)$-dimensional space consists of only one point, so there is at most one non-empty subset we can obtain. Thus, there will be at most two sets in total, and this is achieved for the non-essential set $\{0\} \subseteq \mathbb{R}$.
If $n = 2$, then we are considering a one-dimensional problem. We thus have $\operatorname{cl}\operatorname{conv} A = \operatorname{conv}\operatorname{cl} A$ by Lemma 11, so we can generate at most 4 different non-empty sets, i.e., at most 5 sets in total. On the other hand, for $A = (0,1) \cup (2,3)$ we get the four different non-empty sets A, $\operatorname{cl} A = [0,1] \cup [2,3]$, $\operatorname{conv} A = (0,3)$ and $\operatorname{cl}\operatorname{conv} A = [0,3]$ with the operations cl and conv, so we can generate five different sets from the non-essential set $A = ((0,1) \cup (2,3)) \times \{0\} \subseteq \mathbb{R}^2$.

If $n = 3$, then we have a two-dimensional problem. Now, consider the sets $X_1 = \{(x, y) \in \mathbb{R}^2; y < 1 - 1/(1 + x^2)\}$ and $X_2 = B_{1/4}(0, 3/4)$, and let $A := X_1 \cup X_2$. Then $A = X_1 \cup X_2$, $\operatorname{cl} A = \operatorname{cl} X_1 \cup \operatorname{cl} X_2$, $\operatorname{conv} \operatorname{cl} A = \{(x, y) \in \mathbb{R}^2; y < 1\} \cup \{(0, 1)\}$, $\operatorname{conv} A = \{(x, y) \in \mathbb{R}^2; y < 1\}$ and $\operatorname{cl} \operatorname{conv} A = \{(x, y) \in \mathbb{R}^2; y \leq 1\}$ are five different non-empty subsets of $\mathbb{R}^2$ generated by A with the operations conv and cl. Thus, $A \times \{0\}$ is a non-essential subset of $\mathbb{R}^3$ yielding six different sets.
If $n > 3$, then just take the set $A \subseteq \mathbb{R}^2$ from above, set $O = \{(0, \ldots 0)\} \subseteq \mathbb{R}^{n-2}$ and consider the non-essential set $A \times \{O\} \subseteq \mathbb{R}^n$. From this, we can generate exactly six different sets using cl, int and conv as follows easily from Lemma 12.
Figure 2 shows the set A and the six sets generated by it (if we view it as a subset of the $\mathbb{R}^3$ or a higher-dimensional space).

Proof for the essential case: *Why can't we obtain more than* 17 *sets (*14 *sets for* $n = 1$*)?*
Given an essential set $A \subseteq \mathbb{R}^n$, consider the 17 sets $A_1 := A$, $A_2 := \operatorname{int} A$, $A_3 := \operatorname{cl} \operatorname{int} A$, $A_4 := \operatorname{int} \operatorname{cl} \operatorname{int} A$, $A_5 := \operatorname{conv} \operatorname{cl} \operatorname{int} A$, $A_6 := \operatorname{conv} \operatorname{int} A$, $A_7 := \operatorname{cl} \operatorname{conv} \operatorname{int} A$, $A_8 := \operatorname{cl} A$, $A_9 := \operatorname{int} \operatorname{cl} A$, $A_{10} := \operatorname{cl} \operatorname{int} \operatorname{cl} A$, $A_{11} := \operatorname{conv} \operatorname{cl} \operatorname{int} \operatorname{cl} A$, $A_{12} := \operatorname{conv} \operatorname{int} \operatorname{cl} A$, $A_{13} := \operatorname{cl} \operatorname{conv} \operatorname{int} \operatorname{cl} A$, $A_{14} := \operatorname{conv} \operatorname{cl} A$, $A_{15} := \operatorname{conv} A$, $A_{16} := \operatorname{int} \operatorname{conv} A$ and $A_{17} := \operatorname{cl} \operatorname{conv} A$. We show that every set that we can obtain from A only using the operations int, cl and conv will be one of these 17 sets. It suffices to show that the specified set of sets is closed under the operations int, cl and conv (because it contains A). We will now list all possibilities to apply one of the operations to one of the sets and give the reason why the result will again be one of the given sets:

- $\operatorname{int} A_1 = A_2$, $\operatorname{cl} A_1 = A_8$ and $\operatorname{conv} A_1 = A_{15}$.
- $\operatorname{int} A_2 = \operatorname{int} \operatorname{int} A = \operatorname{int} A = A_2$, $\operatorname{cl} A_2 = A_3$ and $\operatorname{conv} A_2 = A_6$.
- $\operatorname{int} A_3 = A_4$, $\operatorname{cl} A_3 = \operatorname{cl} \operatorname{cl} \operatorname{int} A = \operatorname{cl} \operatorname{int} A = A_3$ and $\operatorname{conv} A_3 = A_5$.
- $\operatorname{int} A_4 = A_4$, $\operatorname{cl} A_4 = \operatorname{cl} \operatorname{int} \operatorname{cl} \operatorname{int} A_4 = \operatorname{cl} \operatorname{int} A_4$ by Lemma 2a as $\operatorname{int} A$ is open, and $\operatorname{conv} A_4 = \operatorname{conv} \operatorname{int} \operatorname{cl} \operatorname{int} A = \operatorname{conv} \operatorname{int} A = A_6$ by Lemma 9.
- $\operatorname{int} A_5 = \operatorname{int} \operatorname{conv} \operatorname{cl} \operatorname{int} A = \operatorname{int} \operatorname{conv} \operatorname{int} A = \operatorname{conv} \operatorname{int} A = A_6$ by Lemma 7 and Lemma 3c, $\operatorname{cl} A_5 = \operatorname{cl} \operatorname{conv} \operatorname{cl} \operatorname{int} A = \operatorname{cl} \operatorname{conv} \operatorname{int} A = A_7$ by Lemma 6 and $\operatorname{conv} A_5 = A_5$.
- $\operatorname{int} A_6 = A_6$ by Lemma 3c, $\operatorname{cl} A_6 = A_7$ and $\operatorname{conv} A_6 = A_6$.

- $\operatorname{int} A_7 = \operatorname{int}\operatorname{cl}\operatorname{conv}\operatorname{int} A = \operatorname{int}\operatorname{conv}\operatorname{int} A = \operatorname{conv}\operatorname{int} A$ by Lemma 8 and Lemma 3c, $\operatorname{cl} A_7 = A_7$ and $\operatorname{conv} A_7 = A_7$ by Lemma 3b.
- $\operatorname{int} A_8 = A_9$, $\operatorname{cl} A_8 = A_8$ and $\operatorname{conv} A_8 = A_{14}$.
- $\operatorname{int} A_9 = A_9$, $\operatorname{cl} A_9 = A_{10}$ and $\operatorname{conv} A_9 = A_{12}$.
- $\operatorname{int} A_{10} = \operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} A = \operatorname{int}\operatorname{cl} A$ by Lemma 2b, $\operatorname{cl} A_{10} = A_{10}$ and $\operatorname{conv} A_{10} = A_{11}$.
- $\operatorname{int} A_{11} = \operatorname{int}\operatorname{conv}\operatorname{cl}\operatorname{int}\operatorname{cl} A = \operatorname{int}\operatorname{conv}\operatorname{int}\operatorname{cl} A = \operatorname{conv}\operatorname{int}\operatorname{cl} A = A_{12}$ by Lemma 7 and Lemma 3c, $\operatorname{cl} A_{11} = \operatorname{cl}\operatorname{conv}\operatorname{cl}\operatorname{int}\operatorname{cl} A = \operatorname{cl}\operatorname{conv}\operatorname{int}\operatorname{cl} A = A_{13}$ by Lemma 6 and $\operatorname{conv} A_{11} = A_{11}$.
- $\operatorname{int} A_{12} = A_{12}$ by Lemma 3c, $\operatorname{cl} A_{12} = A_{13}$ and $\operatorname{conv} A_{12} = A_{12}$.
- $\operatorname{int} A_{13} = \operatorname{int}\operatorname{cl}\operatorname{conv}\operatorname{int}\operatorname{cl} A = \operatorname{int}\operatorname{conv}\operatorname{int}\operatorname{cl} A = \operatorname{conv}\operatorname{int}\operatorname{cl} A = A_{12}$ by Lemma 8 and Lemma 3c, $\operatorname{cl} A_{13} = A_13$ and $\operatorname{conv} A_{13} = A_{13}$ by Lemma 3b.
- $\operatorname{int} A_{14} = \operatorname{int}\operatorname{conv}\operatorname{cl} A = \operatorname{int}\operatorname{conv} A = A_{16}$ by Lemma 7, $\operatorname{cl} A_{14} = \operatorname{cl}\operatorname{conv}\operatorname{cl} A = \operatorname{cl}\operatorname{conv} A = A_{17}$ by Lemma 6 and $\operatorname{conv} A_{14} = A_{14}$.
- $\operatorname{int} A_{15} = A_{16}$, $\operatorname{cl} A_{15} = A_{17}$ and $\operatorname{conv} A_{15} = A_{15}$.
- $\operatorname{int} A_{16} = A_{16}$, $\operatorname{cl} A_{16} = \operatorname{cl}\operatorname{int}\operatorname{conv} A = \operatorname{cl}\operatorname{conv} A = A_{17}$ by Lemma 10 (note that because A is essential, $\operatorname{conv} A$ is also essential, as follows directly from $A \subseteq \operatorname{conv} A$) and $\operatorname{conv} A_{16} = A_{16}$ by Lemma 3a.
- $\operatorname{int} A_{17} = \operatorname{int}\operatorname{cl}\operatorname{conv} A = \operatorname{int}\operatorname{conv} A = A_{16}$ by Lemma 8, $\operatorname{cl} A_{17} = A_{17}$ and $\operatorname{conv} A_{17} = A_{17}$ by Lemma 3b.

For $n = 1$, we have $\operatorname{cl}\operatorname{conv} B = \operatorname{conv}\operatorname{cl} B$ for any $B \subseteq \mathbb{R}$ by Lemma 11, so we get that $A_5 = A_7$, $A_{11} = A_{13}$ and $A_{14} = A_{17}$ and thus always at most 14 sets.
Why can we obtain 17 *sets (*14 *sets for $n = 1$)?* We will first construct a set $A \subseteq \mathbb{R}^2$ such that all the sets $A_1, \dots A_{17}$ as defined above are different.

Let

$$\begin{aligned}
X_1 &:= \left\{(x,y) \in \mathbb{R}^2; x < 0, y > -1, y < 1 - \frac{1}{1-x}\right\},\\
X_2 &:= \left\{(x,y) \in \mathbb{R}^2; x > 0, y > -1, y < 1 - \frac{1}{1+x}\right\} \cap \mathbb{Q}^2,\\
X_3 &:= B_1(0,2) \cap \mathbb{Q}^2,\\
X_4 &:= B_1(-1,-2) \setminus \{-1,-2\},\\
X_5 &:= \{(0,-4)\}
\end{aligned}$$

and

$$A := X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5.$$

We will now state the resulting sets $A_1, \dots A_{17}$ without proofs; in each case, there are very obvious proofs for the correctness of the result. We won't prove either that all the seventeen sets are pairwise distinct, which can also be seen very easily. Figures 3 and 4 show the 17 sets; one can easily see on the figure that they are pairwise different.
For any two real numbers r and s, let $H(r,s) = \{(x,y) \in \mathbb{R}^2; r < y < s\}$. Moreover, note that $\operatorname{cl} X_2 = \left\{(x,y) \in \mathbb{R}^2; x \geq 0, y \geq -1, y \leq 1 - \frac{1}{1+x}\right\}$ and let $\hat{X}_2 = \operatorname{int} \operatorname{cl} X_2 = \left\{(x,y) \in \mathbb{R}^2; x > 0, y > -1, y < 1 - \frac{1}{1+x}\right\}$.
With these notations, we get

- $A_1 = A$
- $A_2 = \operatorname{int} A = X_1 \cup X_4$
- $A_3 = \operatorname{cl} \operatorname{int} A = \operatorname{cl} X_1 \cup \operatorname{cl} X_4$
- $A_4 = \operatorname{int} \operatorname{cl} \operatorname{int} A = X_1 \cup X_4 \cup \{(-1,-2)\}$
- $A_5 = \operatorname{conv} \operatorname{cl} \operatorname{int} A = \operatorname{cl} X_1 \cup \operatorname{cl} X_4 \cup (-\infty, 0] \times [-2,-1] \cup (-\infty, 1) \times (-3,-2)$
- $A_6 = \operatorname{conv} \operatorname{int} A = X_1 \cup B_1(-1,-2) \cup (-\infty, 0) \times [-2,-1] \cup (-\infty, 1) \times (-3,-2)$
- $A_7 = \operatorname{cl} \operatorname{conv} \operatorname{int} A = \operatorname{cl} X_1 \cup \operatorname{cl} X_4 \cup (-\infty, 0] \times [-2,-1] \cup (-\infty, 1) \times [-3,-2)$
- $A_8 = \operatorname{cl} A = \operatorname{cl} X_1 \cup \operatorname{cl} X_2 \cup \operatorname{cl} X_3 \cup \operatorname{cl} X_4 \cup X_5$
- $A_9 = \operatorname{int} \operatorname{cl} A = X_1 \cup \hat{X}_2 \cup \{0\} \times (-1,0) \cup B_1(0,2) \cup B_1(-1,-2)$

- $A_{10} = \operatorname{cl}\operatorname{int}\operatorname{cl} A = \operatorname{cl} X_1 \cup \operatorname{cl} X_2 \cup \operatorname{cl} B_1(0,2) \cup \operatorname{cl} B_1(-1,-2)$
- $A_{11} = \operatorname{conv}\operatorname{cl}\operatorname{int}\operatorname{cl} A = H(-3,3) \cup \{(-1,-3)\} \cup \{(0,3)\}$
- $A_{12} = \operatorname{conv}\operatorname{int}\operatorname{cl} A = H(-3,3)$
- $A_{13} = \operatorname{cl}\operatorname{conv}\operatorname{int}\operatorname{cl} A = \operatorname{cl} H(-3,3)$
- $A_{14} = \operatorname{conv}\operatorname{cl} A = H(-4,3) \cup \{(0,-4)\} \cup \{(0,3)\}$
- $A_{15} = \operatorname{conv} A = H(-4,3) \cup \{(0,-4)\}$
- $A_{16} = \operatorname{int}\operatorname{conv} A = H(-4,3)$
- $A_{17} = \operatorname{cl}\operatorname{conv} A = \operatorname{cl} H(-4,3)$

Figures 3 and Figure 4 show these 17 sets. One easily sees there that they are indeed 17 different sets.
If $n > 2$, then just take the set $A \times \mathbb{R}^{n-2}$ (where A is the set we just described). With the help of Lemma 12, it can easily be seen that this generates 17 different subsets of the $\mathbb{R}^n$ which can be obtained by taking the cartesian products of the 17 sets above with the $\mathbb{R}^n$ — just note that $\operatorname{int}\mathbb{R}^n = \operatorname{cl}\mathbb{R}^n = \operatorname{conv}\mathbb{R}^n = \mathbb{R}^n$.
If $n = 1$, take $A = ((0,1) \cap \mathbb{Q}) \cup (2,3) \cup (3,4) \cup (4,5) \cup \{6\}$. Then

- $A_1 = A = ((0,1) \cap \mathbb{Q}) \cup (2,3) \cup (4,5) \cup (5,6) \cup \{7\}$
- $A_2 = \operatorname{int} A = (2,3) \cup (4,5) \cup (5,6)$
- $A_3 = \operatorname{cl}\operatorname{int} A = [2,3] \cup [4,6]$
- $A_4 = \operatorname{int}\operatorname{cl}\operatorname{int} A = (2,3) \cup (4,6)$
- $A_5 = \operatorname{conv}\operatorname{cl}\operatorname{int} A = [2,6]$
- $A_6 = \operatorname{conv}\operatorname{int} A = (2,6)$
- $A_8 = \operatorname{cl} A = [0,1] \cup [2,3] \cup [4,6] \cup \{7\}$
- $A_9 = \operatorname{int}\operatorname{cl} A = (0,1) \cup (2,3) \cup (4,6)$
- $A_{10} = \operatorname{cl}\operatorname{int}\operatorname{cl} A = [0,1] \cup [2,3] \cup [4,6]$
- $A_{11} = \operatorname{conv}\operatorname{cl}\operatorname{int}\operatorname{cl} A = [0,6]$
- $A_{12} = \operatorname{conv}\operatorname{int}\operatorname{cl} A = (0,6)$

- $A_{14} = \operatorname{conv} \operatorname{cl} A = [0, 7]$
- $A_{15} = \operatorname{conv} A = (0, 7]$
- $A_{16} = \operatorname{int} \operatorname{conv} A = (0, 7)$

so that A generates 14 different sets.

Appendix: Figures for the given examples

In this appendix, we give three figures showing example sets described in the above solutions to prove several existence claims: Figure 1 shows the set from Problem 9.1 from which one can obtain 7 sets only using the operations int and cl (the set is shown for $n = 2$; it is easy to verify and to imagine that the same construction also works for $n = 1$ as well as for every bigger value of n); Figure 2 shows the non-essential set from which one can obtain 6 sets by $\operatorname{int}, \operatorname{cl}$ and conv (shown for $n = 3$; the set for $n = 2$ is a subset of a two-dimensional hyperplane in $\mathbb{R}^3$ so that it can indeed be drawn on a piece of paper; the example for higher dimensions is the same, only that it is embedded into an higher-dimensional space; the example for $n = 1$ is not shown); and Figures 3 and 4 show the essential set from problem 9.2 from which one can obtain 17 sets with the operations $\operatorname{int}, \operatorname{cl}$ and conv (also shown for $n = 2$; the example in higher dimensions will consist of several copies of this, one for each point of the $\mathbb{R}^{n-2}$; the example for $n = 1$ is not shown). In all the examples, a part of the boundary of a set which is drawn 'fat' indicates that in this part of the boundary, the boundary points belong to the set.

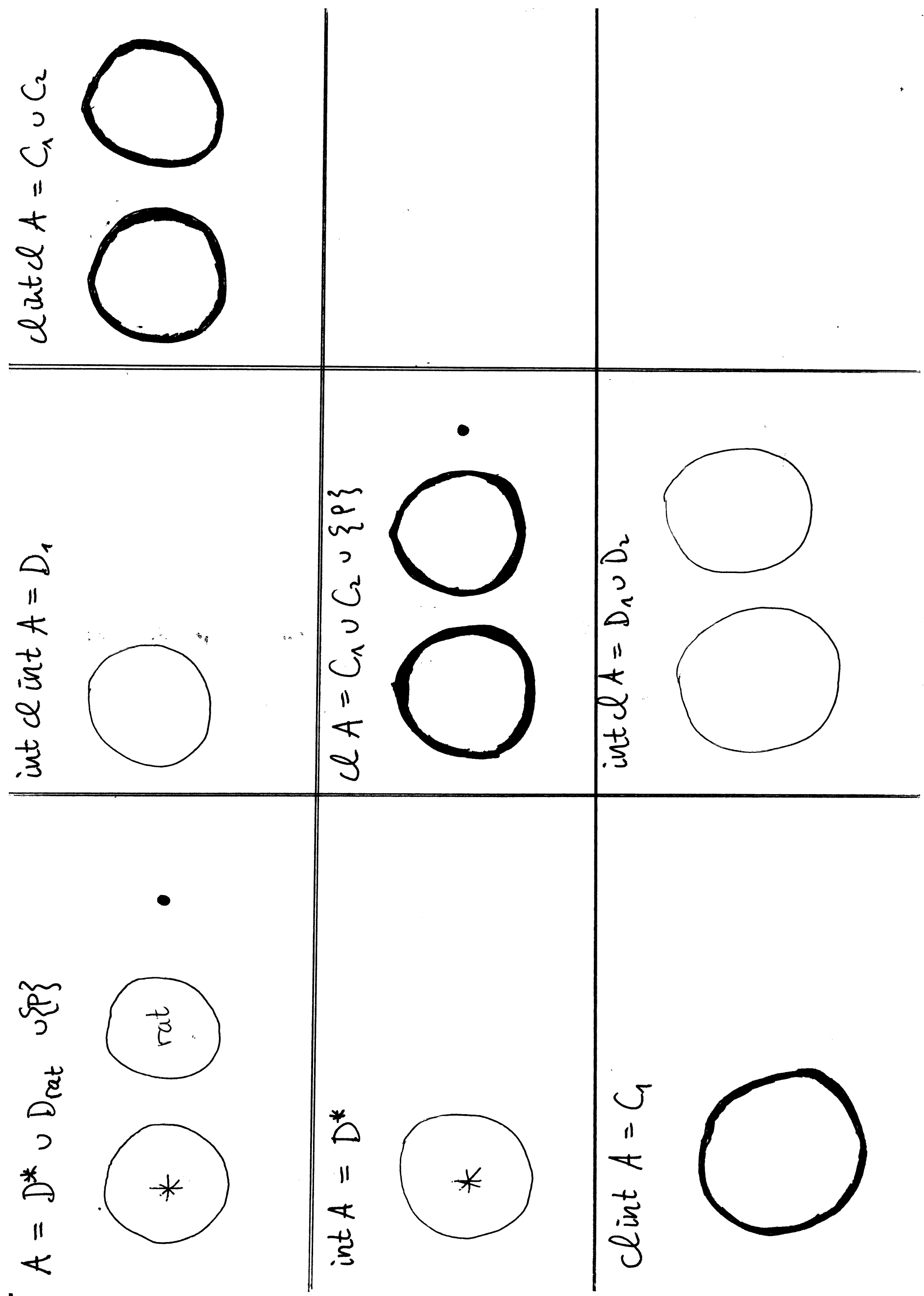

Figure 1: Example set for Problem 9.1

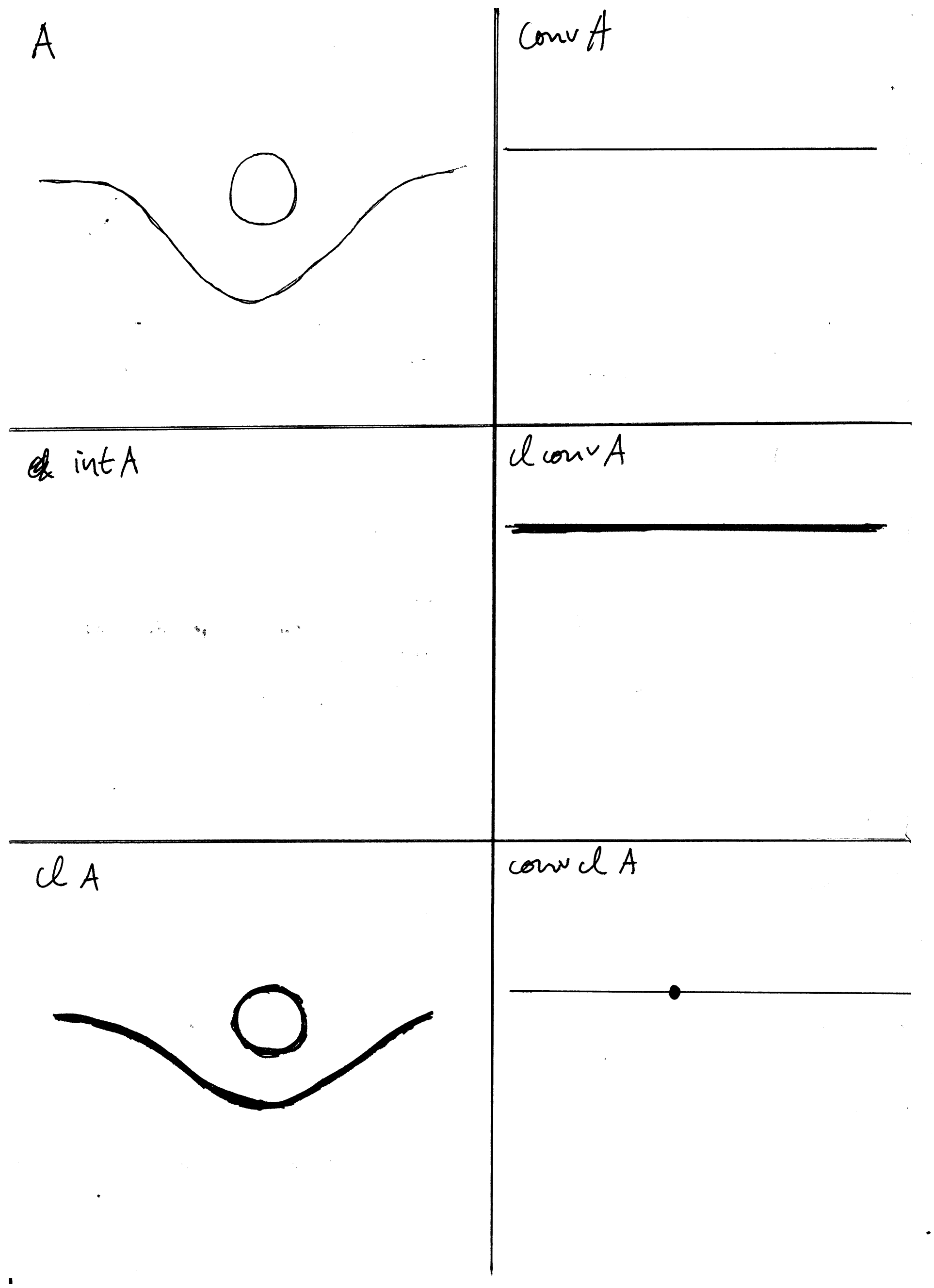

Figure 2: Non-essential example set for Problem 9.2

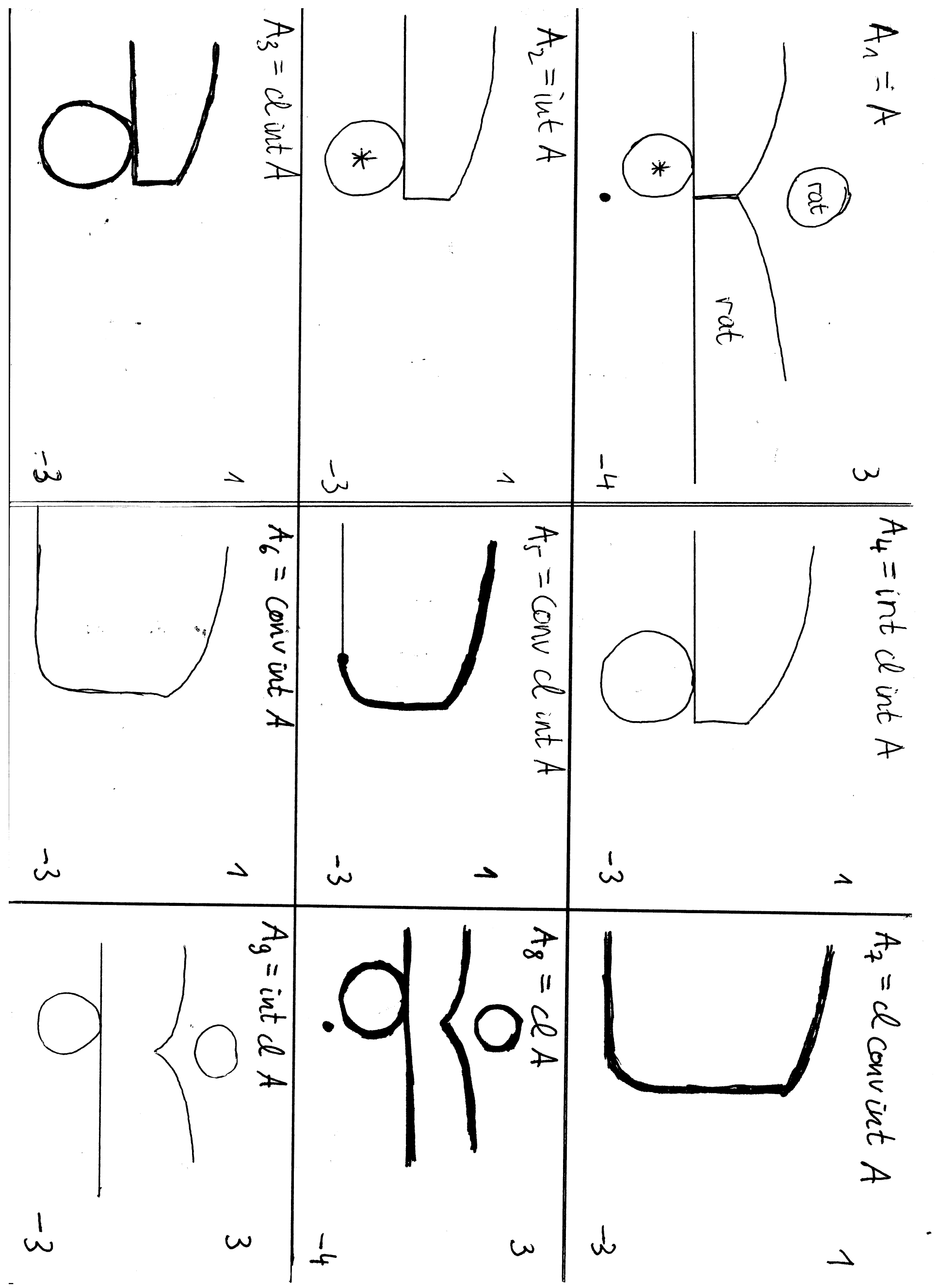

Figure 3: Essential example set for Problem 9.2, first part

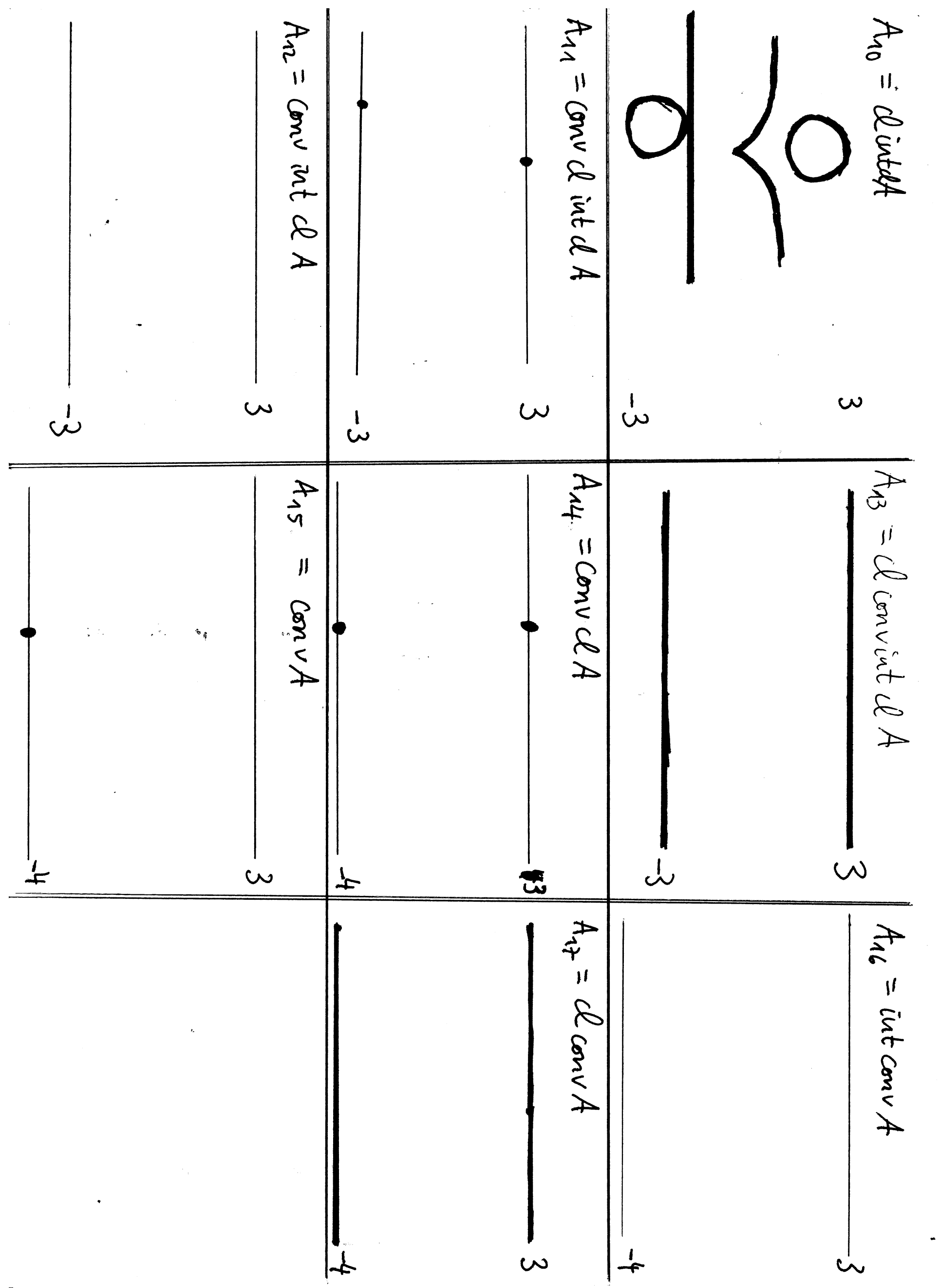

Figure 4: Essential example set for Problem 9.2, second part