Problem 8: Points on Curves

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Abstract

We didn't manage to solve any part of the problem, but we give a short overview about the main ideas that we have had on Problem 8.1 (although none of them has lead to a solution).

The best result we have achieved is that we always can find points $t_0, \ldots, t_n$ as requested in the problem statement if f is monotonous or satisfies $|f(x) - f(y)| < |x - y|$ for all x, y.

We first introduce some notation: For any two points $a, b \in [0,1]$, let $d(a,b) = |a-b| + |f(a) - f(b)|$. It is easy to see that $d(a,b) = 0$ if and only if $a = b$, that $d(a,b) = d(b,a)$ and that $d(a,b) + d(b,c) \geq d(a,c)$. Moreover, if we fix one of a, b and view $d(a,b)$ as a function in the other variable, then this function will be continuous (as f is).

Special Cases

$n = 1$ and $n = 2$

For $n = 1$ and $n = 2$, there will always be numbers satisfying the conditions of the problem. This is very easy to see: For $n = 1$, it's totally obvious; for $n = 2$, define a function $g : [0,1] \to \mathbb{R}, x \mapsto d(0,x) - d(1,x)$. This function is continuous in x from what we said above, and we have $g(0) = -d(0,1) < 0$, but $g(1) = d(1,0) > 0$. Thus, by the intermediate value theorem, there is a number $t_1 \in [0,1]$ such that $g(t_1) = 0$, implying that $d(t_1, 0) = d(t_1, 1)$ or

$$|f(t_1) - f(0)| + |t_1 - 0| = |f(1) - f(t_1)| + |1 - t_1|.$$

Thus, by setting $t_0 = 0, t_2 = 1$, we have found a sequence as wished in the problem.

Monotonous Functions

We have been able to prove the following fact:

Theorem 1. If f is strictly monotonically increasing (or decreasing), then there is a sequence $(t_0, \ldots, t_i)$ with the properties described in the problem statement for every n.

This is part of the following more general theorem:

Theorem 2. Suppose that for every $a \in [0,1]$, the function $x \mapsto d(x,a)$ is monotonically decreasing on $[0,a]$ and increasing on $[a,1]$. Then there is a sequence of points $t_0, t_1, \ldots, t_n$ with the properties required in the problem statement for any natural number n.

Proof. For formal reasons, we define f not only on $[0,1]$, but on $\mathbb{R}_{\geq 0}$ by setting $f(x) := f(1)$ for $x \geq 1$. Let s be any nonnegative real number. Set $t_0 = t_0(s) = 0$. Then there will be exactly one nonnegative real number $t_1 = t_1(s)$ such that $t_0 < t_1$ and $d(t_0, t_1) = s$. The existence of such a number follows from the continuity of $d(t_0, x)$ as a function of x and the fact that we have $d(t_0, t_0) = 0$ and $d(t_0, x) > |x - t_0|$, so that $d(x, t_0)$ is unbounded on $\mathbb{R}_{\geq 0}$. The uniqueness follows from the fact that $d(x, t_0)$ grows strictly monotonously. Note that the function mapping s to $t_1(s)$ is continuous. This can be shown as follows: Suppose that it's not continuous. Then there is a point $s \in [0, 1]$ and a sequence (s_i) converging to s such that $|t_1(s_i) - t_1(s)| > \varepsilon$ for some positive real number ε and all i. Infinitely many of the s_i have to lie on one side of s: Suppose that infinitely many are greater than s. Equivalently, we can suppose that all s_i are greater than s. Then we have $t_1(s_i) > t_1(s) + \varepsilon$ for all i, so it follows that

$$s_i = d(t_0, t_1(s_i)) > d(t_0, t_1(s) + \varepsilon)$$

for all i. But $d(t_0, t_1(s) + \varepsilon)$ is a real number greater than $d(t_0, t_1(s)) = s$, so the inequality above contradicts the assumption that $s_i \to s$. Thus, $s \mapsto t_1(s)$ is continuous.
Similarly, there exist points $t_2(s), \ldots, t_n(s) \in \mathbb{R}_{\geq 0}$ such that we always have $t_i(s) < t_{i+1}(s)$ and $d(t_i(s), t_{i+1}(s)) = s$, and all the $t_i(s)$ are continuous functions of s; all of this is shown very similarly as above.
Now, we surely have $t_n(0) = 0$ and we have to have $t_n(s) > 1$ for very large values of s (for example, for $s > d(0, 1)$, we get $t_n(s) > t_1(s) > 1$), so by the intermediate value theorem, there has to be a number s such that $t_n(s) = 1$. But then

$$d(t_0(s), t_1(s)) = \ldots = d(t_{n-1}(s), t_n(s)) = s,$$

so we are done. □

Note that the condition of Theorem 1 is satisfied if f grows monotonously (then we have $d(x, a) = f(x) - f(a) + x - a$ for $x \geq a$, which obviously is strictly increasing, and $d(x, a) = f(a) - f(x) + a - x$ for $x \leq a$, which is obviously monotonously decreasing), or monotonically decreasing (because then $-$ is monotonically increasing) or if $|f(x) - f(y)| < |x - y|$ for all f: In the latter case, we have for $x > y > a$:

$$\begin{aligned} d(x, a) - d(y, a) &= x - a + |f(x) - f(a)| - (y - a + |f(y) - f(a)|) \\ &\geq x - y - |f(y) - f(x)| = |x - y| - |f(x) - f(y)| > 0. \end{aligned}$$

(The fact that $d(x, a)$ is strictly monotonically decreasing is shown similarly.) Thus, for those two types of functions, there always exist sequences as required.

Main Ideas

In the following, we list our main ideas to solve this problems, but also reasons why we think they can't work.

- If $d(x, y)$ is not defined by the formula above, but is the usual distance in the plane, then one possible translation of our problem is as follows: Given some continuous curve in the plane and some natural number n, prove that it's possible to divide the curve into n parts of equal lengths. This statement is (true and) rather easy to prove, so we tried to imitate proofs of this statement.

- As $[0, 1]$ is compact, f has to be uniformly continuous, so it suffices to consider a "discretized version" of the problem: If f is defined on the set $\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\}$ and we have $f((i+1)/N) - f(i/N) < \varepsilon$ for any i, then show that we can find indices $i_1 < i_2 < \ldots < i_n$ such that the values of $|(i_{j+1} - i_j)/N| + |f(i_{j+1}/N) - f(i_j/N)|$ differ by at most $G(\varepsilon)$, where $G(\varepsilon) \to 0$ when $\varepsilon \to 0$.

- One could try to do some kind of induction or also define points recursively starting in $(0, f(0))$ such that the distance between two adjacent points is constant (as done in the proof of Theorem 1). The problem with generalizing this proof is that a point x is, in general, not defined uniquely by another point y and the distance $d(x, y)$, so if we want to define such points as a function of a starting point, we have to choose the point somehow (take the smallest or the biggest or the nearest point...) and then the function won't be continuous any more in general.

- As can also be seen from the above points, we believe that one can *always* find a sequence $t_0, t_1 \ldots t_n$ with the requested properties (for any n). We also proved this for a number of special cases (for example, for a polygonal chain consisting of two segments and $n = 3$) by brute force calculations.