Problem 6: Min/Max Questions

Team Germany

Abstract

In the following we establish some of the Min / Max Questions, mainly focussing on 6.1. In detail, we give exact upper bounds of $l_{\min}/l_{\max}$ for the special cases mentioned in the problems a) and b). Furthermore, we show a more general inequality for the case that S is the whole plane (f) and explain the connection between part e) and f) of 6.1. In part 6.2, we give complete solutions of the problems a), b) and c) by using simple arguments based on the extremal principle.

Problem 6.1

In addition we state that it is obvious that the ratio $l_{\min}/l_{\max}$ can receive all enough small values.
a) If S is a line, the situation is nearly trivial. We name the N points from the left to the right by P_1 to P_N. Then it is obviously that $\overline{P_1P_N} = l_{\max}$. Denote by j an index for which $\overline{P_jP_{j+1}} = l_{\min}$. Now we have:

$$l_{\max} = \overline{P_1P_N} = \sum_{i=1}^{N-1} \overline{P_iP_{i+1}} \geq (N-1)\cdot \overline{P_jP_{j+1}} = (N-1)\cdot l_{\min}$$

b) We state that $l_{\min}/l_{\max} \leq \sin(\frac{\pi}{N})$ if N is even and $l_{\min}/l_{\max} \leq 2\sin(\frac{\pi}{2N})$ if N is odd.

First case: $N = 2k$ is even. The case $k = 1$ is trivial, so we assume $k \geq 2$. Consider N arbitrary points on a circumference the diameter of which can be assumed to be 1. Let the points be $P_1, \ldots, P_N$, such that each of the arcs P_iP_{i+1} ($i = 1, \ldots, n$, $P_{n+1} := P_1$) does not contain any further point P_j. W.l.o.g., the points $P_2, \ldots, P_k$ and the center O of the circumference are located on different sides of P_1P_{k+1}. Define $\varphi := 1/2\angle P_1OP_{k+1}$. We then have $l_{\max} \geq P_1P_{k+1} = \sin(\varphi)$. Furthermore, at least one of the angles $\angle P_iOP_{i+1}$, $i = 1, \ldots, k$ is not bigger than φ/k. Using the monotonicity of $x \mapsto \sin(x)$ in $[0, \pi/2]$ we thus get

$$l_{\min} \leq \min\{P_iP_{i+1} | i = 1, \ldots, k\} = \min\{\sin(\angle P_iOP_{i+1}) | i = 1, \ldots, k\} \leq \sin(\varphi/k).$$

In conclusion, we have $l_{\min}/l_{\max} \leq \sin(\varphi/k)/\sin(\varphi)$. An easy calculation shows that the function $x \mapsto \sin(x/k)/\sin(x)$ is monotonous increasing in $[0, \pi/2]$. From this,

$$l_{\min}/l_{\max} \leq \sin(\frac{\pi}{2k})/\sin(\frac{\pi}{2}) = \sin(\frac{\pi}{N})$$

follows. Inversely, the value $\sin(\frac{\pi}{N})$ can be attained by choosing the N points as vertices of a regular N-gone.

Second case: $N = 2k-1$ is odd. Like above, let the points be $P_1, \ldots, P_N$, such that each of the arcs P_iP_{i+1} ($i = 1, \ldots, n$, $P_{n+1} := P_1$) does not contain any further point P_j. We can assume that $l_{\min} = P_kP_{k+1}$. If the center O of the circumference is not located inside the triangle $P_1P_kP_{k+1}$, the same argument as above shows that $l_{\min}/l_{\max} \leq \sin(\frac{\pi}{2k})$. Now, we can prove that $\sin(\frac{\pi}{2k}) \leq 2\sin(\frac{\pi}{2N})$ by calculus: The derivate of the function $x \mapsto \sin(\frac{\pi}{2x}) - 2\sin(\frac{\pi}{4x-2})$, $x \geq 1$,

$$f'(x) = -\frac{1}{2x^2}\cos(\frac{\pi}{2x}) + \frac{8}{(4x-2)^2}\cos(\frac{\pi}{4x-2}) = \frac{1}{(2x-1)(x-\frac{1}{2})}\cos(\frac{\pi}{4x-2}) - \frac{1}{2x^2}\cos(\frac{\pi}{2x})$$

is positive for alle $x \geq 1$, and the function itself tends to 0 for $x \to \infty$.

Thus, we need only to consider the case that O is inside the triangle $P_1P_kP_{k+1}$. Let φ be the smaller of the angles $1/2\angle P_1OP_k$ and $1/2\angle P_{k+1}OP_1$. Because the segment P_kP_{k+1} was minimal, we have $(2 + \frac{1}{k-1})\varphi \leq \pi$, that is $\varphi \leq \frac{k-1}{2k-1}\pi$. The same argumentation as above shows now that

$$l_{\min}/l_{\max} \leq \frac{\sin(\frac{\varphi}{k-1})}{\sin(\varphi)} \leq \frac{\sin(\pi\frac{k-1}{(k-1)(2k-1)})}{\sin(\pi\frac{k-1}{2k-1})} = \frac{\sin(\frac{\pi}{N})}{\sin(\frac{\pi}{2}-\frac{\pi}{2N})} = \frac{\sin(\frac{\pi}{N})}{\cos(\frac{\pi}{2N})} = 2\sin(\frac{\pi}{2N}).$$

Inversely, we can easily show that the value $2\sin(\frac{\pi}{2N})$ is attainable by choosing the N points as vertices of a regular N-gon.
e) The answer is the same as for problem f), because every configuration of N points in the plane can be arbitrarily good approximated by a configuration of N points on the infinite triangular grid. More precisely: It is obvious that the maximal value of $l_{\min}/l_{\max}$ is smaller for the grid than for the plane. On the other hand, if we have a configuration in the plane such that the ratio $l_{\min}/l_{\max}$ is maximal (or nearly maximal), then we can take an infinite triangular grid with very small triangles, put it into the same plane and substitute the points from the configuration by the nearest points in the grid. If we make the grid small enough, then the position of the points will change by nearly nothing, so the change in $l_{\min}/l_{\max}$ can also be made arbitrarily small.
f) For $N = 4$: If the convex hull of the four points is a triangle, there is one point in the interior or at the boundary of that convex hull. Then one of the angles at this point between two of the other points will be at least $\frac{2\pi}{3} > \frac{\pi}{2}$. If the convex hull is a quadrilateral, one of the internal angles will be at least $\frac{\pi}{2}$. So in every case there is a point Q with two other points Q_1 and Q_2 such that $\angle Q_1QQ_2 = \varphi \geq \frac{\pi}{2}$. We denote $\overline{QQ_1} = l_1$ and $\overline{QQ_2} = l_2$. With the cosine rule we obtain:

$$l_{\max}^2 \geq \overline{Q_1Q_2}^2 = l_1^2 + l_2^2 - 2\cdot l_1 l_2 \cos(\varphi) \geq l_1^2 + l_2^2 \geq 2l_{\min}^2$$

and hence

$$l_{\max} \geq \sqrt{2}\cdot l_{\min}.$$

We prove now a general theorem: if we have N points in the plane the following inequality holds

$$l_{\max} \geq \frac{\sqrt{3}}{2}(\sqrt{N}-1)\cdot l_{\min}$$

Set $D = l_{\max}$ and $d = l_{\min}$. Consider around each of the N points a cercle with radius $\frac{d}{2}$. Refering to the Theorem of Jung we know that all N points lie in a circle with radius $\frac{D}{\sqrt{3}}$. So all circles around the N points lie in a circle with radius $\frac{D}{\sqrt{3}} + \frac{d}{2}$. Now, we can estimate the areas of the little circles and the area of the big circle:

$$N\pi(d/2)^2 \leq \pi\left(\frac{D}{\sqrt{3}} + \frac{d}{2}\right)^2$$

$$\sqrt{N}\cdot\frac{d}{2} \leq \frac{D}{\sqrt{3}} + \frac{d}{2}$$

$$D \geq \frac{\sqrt{3}}{2}(\sqrt{N}-1)\cdot d.$$

Problem 6.2

a) Is the same as problem 6.1a, as lines are convex.
b) The ratio $c_{\min}/c_{\max}$ can take every value in $]0, \frac{1}{[\frac{N}{2}]}]$. It is enough to show that the inequality $c_{\min}/c_{\max}\frac{1}{[\frac{N}{2}]}$ holds and to prove that equality can be attained. Let the points be $P_1, \dots, P_N$, such that each of the arcs P_iP_{i+1} ($i = 1, \dots, n$, $P_{n+1} := P_1$) does not contain any further point P_j. The smaller of the two arcs $P_1P_{[\frac{N}{2}]+1}$ is not greater than $c_{\max}$ and consists of at least $[\frac{N}{2}]$ arcs P_iP_{i+1} each of which has at least the length $c_{\min}$. From this we get $c_{\max} \geq \widehat{P_1P}_{[\frac{N}{2}]+1} \geq [\frac{N}{2}]c_{\min}$ which implies our claim.
c) This is exactly the same as b), because there exists a function f which maps the polygon on a circle and preserves curve distances (up to some constant factor).
f) This is the same as problem 6.1f, as the plane is convex.

Problem 6.3

a) This is senseless: Any triangle the vertices of which lie on a line has area 0.